FORGE

Fate Of Repository Gases

European Commission FP7

BASIC FORMULATION FOR MODELLING WATER AND GAS FLOW THROUGH THE EDZ. APPLICATION TO HG-A IN SITU TEST

FORGE Report D4.12

	Name	Organisation	Signature	Date
Compiled	J. Alcoverro	UPC		June 2013
Verified	RP Shaw	BGS		14th October 2013
Approved	RP Shaw	BGS		14th October 2013

Keywords.

Lasgit; bentonite; Gas flow; modelling

Bibliographical reference

J. ALCOVERRO, S. OLIVELLA, AND E.E. ALONSO. 2013. BASIC FORMULATION FOR MODELLING WATER AND GAS FLOW THROUGH THE EDZ. APPLICATION TO HG-A IN SITU TEST

FORGE Report D4.12. 44pp.

Euratom 7th Framework Programme Project: FORGE

EURATOM

Fate of repository gases (FORGE)

The multiple barrier concept is the cornerstone of all proposed schemes for underground disposal of radioactive wastes. The concept invokes a series of barriers, both engineered and natural, between the waste and the surface. Achieving this concept is the primary objective of all disposal programmes, from site appraisal and characterisation to repository design and construction. However, the performance of the repository as a whole (waste, buffer, engineering disturbed zone, host rock), and in particular its gas transport properties, are still poorly understood. Issues still to be adequately examined that relate to understanding basic processes include: dilational versus visco-capillary flow mechanisms; long-term integrity of seals, in particular gas flow along contacts; role of the EDZ as a conduit for preferential flow; laboratory to field up-scaling. Understanding gas generation and migration is thus vital in the quantitative assessment of repositories and is the focus of the research in this integrated, multi-disciplinary project. The FORGE project is a pan-European project with links to international radioactive waste management organisations, regulators and academia, specifically designed to tackle the key research issues associated with the generation and movement of repository gasses. Of particular importance are the long-term performance of bentonite buffers, plastic clays, indurated mudrocks and crystalline formations. Further experimental data are required to reduce uncertainty relating to the quantitative treatment of gas in performance assessment. FORGE will address these issues through a series of laboratory and field-scale experiments, including the development of new methods for up-scaling allowing the optimisation of concepts through detailed scenario analysis. The FORGE partners are committed to training and CPD through a broad portfolio of training opportunities and initiatives which form a significant part of the project.

Further details on the FORGE project and its outcomes can be accessed at www.FORGEproject.org.

Contact details:

J. Alcoverro, S. Olivella, E.E. Alonso

Department of Geotechnical Engineering and Geosciences

Universitat Politècnica de Catalunya

Jordi Girona 1-3

08034 Barcelona

Spain

BASIC FORMULATION FOR MODELLING WATER AND GAS FLOW THROUGH THE EDZ. APPLICATION TO HG-A IN SITU TEST

D4.12 - Progress report – definition of constitutive laws and initial simulations

(Rev 1.03)

J. Alcoverro, S. Olivella, E.E. Alonso

Department of Geotechnical Engineering and Geosciences
Universitat Politècnica de Catalunya
Jordi Girona 1-3, 08034 Barcelona

TABLE OF CONTENTS

1 Introduction 3
2 Modelling approach 4
2.1 Balance equations for a porous material with two fluid phases 4
2.2 Equilibrium restrictions 6
2.3 Constitutive relations 7
2.4 Field equations 11
2.5 Transition between saturated and unsaturated states 13
3 1D Modelling 15
3.1 Geometry and coordinate system 15
3.2 Simplifying assumptions 16
3.3 Consequences of the simplifying assumptions 16
3.4 Field equations, initial and boundary conditions 19
3.5 Test input protocol 22
4 Numerical solution using the FEM 25
4.1 Weak form of the problem 25
4.2 Reduction to a system of ordinary differential equations 28
4.3 Integration of the system of ordinary differential equations 31
4.4 Solution of the systems of algebraic equations 33
4.5 Time marching scheme 36
5 Preliminary results 38
5.1 Verification of the simplifying assumptions 38
5.2 Predicted evolutions with time of some important variables 40
6 Conclusions 43
7 References 44

APPENDICES

A1. Some functions and their derivatives with respect to p_l and p_g 46
A1.1. Selected functions of p_l and p_g 47
A1.2. First order derivatives of the selected functions with respect to p_l and p_g 51
A2. Isotropic Linear Elasticity in Plain Strain and Axisymmetry 56
A2.1. Compressible Material ($-1<\nu<1/2$) 57

A2.2. Incompressible Material ($\nu = 1/2$) 58
A3. Isotropic Linear Poroelasticity in Plain Strain and Axisymmetry 61
A3.1. Drained conditions with constant p^* 61
A3.2. Undrained conditions 62
A4. 1D Modelling of the HG-A Test 64
A4.1. Solution on planes z = const 64
A4.2. Influence of the effective parameters of the Opalinus clay 67
A5. Weak form of the problem 68
A5.1. Balance of water mass 68
A5.2. Balance of air mass 70
A6. Shape functions, their derivatives and some related integrals 72
A6.1. Shape functions 72
A6.2. Derivatives of the shape functions 72
A6.3. Integrals of the shape functions and of their binary products 73
A7. Nodal Equations 74
A7.1. Balance of water mass 74
A7.2. Balance of air mass 75
A8. Element and node matrices and vectors 76
A8.1. Matrix $\boldsymbol{C}_{[z^I, z^{I+1}]}$ 76
A8.2. Matrix $\boldsymbol{K}_{[z^I, z^{I+1}]}$ 77
A8.3. Vector $\boldsymbol{g}_{[z^I, z^{I+1}]}$ 79
A8.4. Matrix $\boldsymbol{K}_{z^N}$ 79
A8.5. Vector $\boldsymbol{g}_{z^1}$ 79
A8.6. Vector $\boldsymbol{g}_{z^N}$ 79

1 Introduction

The HG-A test is located in the Mont Terri Rock Laboratory, near Saint Ursanne (Jura, Switzerland). It consists of a horizontal borehole of 1.00 m of diameter and 13.00 m of length excavated in the ultra-low permeable Opalinus clay. During the tunnel drilling, the Opalinus clay near the tunnel wall was damaged, giving rise to an EDZ (Excavation Damaged Zone) around the tunnel. After drilling the tunnel, a steel liner was placed along the 6.00 m close to the tunnel mouth (the liner section) in order to guarantee the tunnel stability. Then, the last 4.00 m at the tunnel end (the test section) were backfilled with gravel. Finally, along the remaining 3.00 m (the megapacker section), an inflatable rubber packer, similar to those used to isolate intervals in a borehole but of 1.00 m in diameter (the megapacker), was installed and inflated, thereby compressing the EDZ that was created during the tunnel excavation. The test section was filled with de-aired water and care was taken in order to eliminate the air from this tunnel section. Subsequently, a series of water and gas injection tests were carried out with varying megapacker pressure, whereby water or gas was injected into the test section and, due to the very low permeability of the intact Opalinus clay, forced to flow back along the EDZ. A description of the HG-A in situ test, including installation and operation, a characterisation of the EDZ induced around the tunnel and results obtained from this test may be found in Marschall et al. (2006), Marschall et al. (2008) and Lanyon et al. (2008). Gas transport properties of the Opalinus clay may are presented in Marschall et al. (2005). Details of the instrumentation of the HG-A in-situ test are given in Trick et al. (2007).

The formation of the EDZ due to the tunnel excavation has been previously modeled using a 3D finite numerical model using code_bright (a finite element code developed at the Geotechnical Engineering Department of the UPC), whereby an anisotropic linear elastic material coupled with a constant anisotropic intrinsic permeability was used to model the Opalinus clay. In order to model the water and gas flow through the EDZ, we have followed a two-track approach. On the one hand, a 2D axisymmetric numerical model using code_bright has been made. On the other hand, a 1D analytical-numerical model has been developed and implemented in an Excel spreadsheet, whereby the field equations defined on a 1D geometrical domain are numerically solved using the finite element method. The 1D model has been used in order to calibrate the 2D axisymmetric model. In this report, the 1D model will be described in detail, and some of their results will be shown.

It should be mentioned that, from the point of view of the formulation leading to the field equations, the differences between the 1D model and the 2D axisymmetric model using code_bright are essentially the simplifying assumptions whereas, from the numerical point of view, the numerical solution of the 1D model follows more closely the standard Galerkin approach that is frequently used in the Finite Element Method.

2 Modelling approach

The modelling approach in based on Olivella et al. (1994). Both the Opalinus clay and the EDZ will be considered to be porous media, with a solid phase (with one species: clay), a liquid phase (with two species: water and air) and a gas phase (with two species: water and air). In principle, balance of mass, momentum and energy should be established for the various species in the various phases present in the porous medium. However, equilibrium restrictions and constitutive relations will allow dropping some of them.

Exchanges of both species water and species air between the liquid phase and the gas phase will be allowed. However, it will be assumed that they are always in equilibrium. It will be assumed that motions are slow so that terms involving accelerations and products of velocities may be neglected. Finally, it will be assumed that temperature remains constant everywhere.

By combining the balance equations, the equilibrium restrictions and the constitutive relations, we get the field equations with the displacement vector of the solid phase $\boldsymbol{u}_s$, the pressure of the liquid phase p_l and the pressure of the gas phase p_g as unknown functions of the space position and time.

2.1 Balance equations for a porous material with two fluid phases

Balance equations are general equations (independent of constitutive assumptions) stating fundamental physical principles concerning mass, momentum and energy

- **Balance of mass**

As explained later, due to the assumption of phase change equilibrium, the balance of mass needs only be considered for the solid phase, for the species water and for the species air

$$\frac{\partial}{\partial t}\{\rho_s(1-\phi)\}+div\{\rho_s(1-\phi)\boldsymbol{v}_s\}=0 \tag{2.1}$$

$$\frac{\partial}{\partial t}\{\omega_l^w\rho_l S_l\phi+\omega_g^w\rho_g S_g\phi\}+div\{\boldsymbol{i}_l^w+\omega_l^w\rho_l\boldsymbol{q}_l+\boldsymbol{i}_g^w+\omega_g^w\rho_g\boldsymbol{q}_g+(\omega_g^w S_g\rho_g+\omega_l^w S_l\rho_l)\phi\boldsymbol{v}_s\}=0 \tag{2.2}$$

$$\frac{\partial}{\partial t}\{\omega_l^a\rho_l S_l\phi+\omega_g^a\rho_g S_g\phi\}+div\{\boldsymbol{i}_l^a+\omega_l^a\rho_l\boldsymbol{q}_l+\boldsymbol{i}_g^a+\omega_g^a\rho_g\boldsymbol{q}_g+(\omega_g^a S_g\rho_g+\omega_l^a S_l\rho_l)\phi\boldsymbol{v}_s\}=0 \tag{2.3}$$

where ρ_s is the density of the solid phase, ρ_l is the density of the liquid phase, ρ_g is the density of the gas phase, ϕ is the void ratio, S_l and S_g are the degree of saturation of the liquid phase and of the gas phase, $\boldsymbol{v}_s$ is the velocity of the solid phase, ω_l^w and ω_l^a are the mass fractions of species water and of species air in the liquid phase, ω_g^w and ω_g^a are the mass fractions of species water and of species air in the gas phase, $\boldsymbol{q}_l$ and $\boldsymbol{q}_g$ are the volume fluxes of the liquid phase and of the gas phase with respect of the solid phase ($\boldsymbol{q}_l=\phi S_l(\boldsymbol{v}_l-\boldsymbol{v}_s)$ and $\boldsymbol{q}_g=\phi S_g(\boldsymbol{v}_g-\boldsymbol{v}_s)$, where $\boldsymbol{v}_l$ and $\boldsymbol{v}_g$ are the velocities of the liquid phase and of the gas phase), $\boldsymbol{i}_l^w$ and $\boldsymbol{i}_l^a$ are the mass fluxes with respect to the liquid phase of the species water in

the liquid phase and of the species air in the liquid phase ($\boldsymbol{i}_l^w = \rho_l \omega_l^w \phi S_l (\boldsymbol{v}_l^w - \boldsymbol{v}_l)$ and $\boldsymbol{i}_l^a = \rho_l \omega_l^a \phi S_l (\boldsymbol{v}_l^a - \boldsymbol{v}_l)$, where $\boldsymbol{v}_l^w$ and $\boldsymbol{v}_l^a$ are the velocities of the species water in the liquid phase and of the species air in the liquid phase) and $\boldsymbol{i}_g^w$ and $\boldsymbol{i}_g^a$ are the mass fluxes with respect to the gas phase of the species water in the gas phase and of the species air in the gas phase ($\boldsymbol{i}_g^w = \rho_g \omega_g^w \phi S_g (\boldsymbol{v}_g^w - \boldsymbol{v}_g)$ and $\boldsymbol{i}_g^a = \rho_g \omega_g^a \phi S_g (\boldsymbol{v}_g^a - \boldsymbol{v}_g)$, where $\boldsymbol{v}_g^w$ and $\boldsymbol{v}_g^a$ are the velocities of the species water in the liquid phase and of the species air in the liquid phase).

The balance of mass of the solid phase (2.1) may be integrated, obtaining

$$J^s \rho_s (1-\phi) = \rho_s^0 (1-\phi^0), \tag{2.4}$$

where J^s is the Jacobian of the motion of the solid phase, ρ_s^0 is the density of the solid phase at the corresponding point in the reference configuration and ϕ^0 is the porosity at the corresponding point in the reference configuration. Using the relation (small strains)

$$J_s = 1 + \varepsilon_{s,vol} \tag{2.5}$$

we arrive to

$$(1+\varepsilon_{s,vol}) \rho_s (1-\phi) = \rho_s^0 (1-\phi^0). \tag{2.6}$$

Because of the assumption of equilibrium of the exchange of the species water and of the species air between the liquid phase and the gas phase, the balance of mass for the species water in the liquid phase, the balance of mass for the species air in the liquid phase, the balance of mass for the species water in the gas phase and the balance of mass for the species air in the gas phase are not needed. However, they allow calculating the mass rates of exchange of the species water and of the species air between the liquid phase and the gas phase, as shown below.

- **Balance of momentum**

If motions are slow, then the balance of momentum for the porous medium as a whole reads

$$div\, \boldsymbol{\sigma} + \rho \boldsymbol{g} = \boldsymbol{0}, \tag{2.7}$$

where $\boldsymbol{\sigma}$ is the (total) stress tensor of the porous medium, ρ is the (total) density of the porous medium ($\rho = (1-\phi)\rho_s + \phi S_l \rho_l + \phi S_g \rho_g$) and $\boldsymbol{g}$ is the gravity acceleration .

The balance of momentum for the liquid and the gas phases will be replaced by generalized Darcy's laws appropriate for unsaturated materials. These laws correspond to the balance of momentum for the liquid phase and for the gas phase and to constitutive assumptions for the stress tensors of each of these phases and for the exchange of momentum of each of these phases with the other phases. They are presented with the constitutive relations. The balance of momentum of the solid phase may be obtained by noting that the sum of the balance of momentum for the solid phase, for the liquid phase and for the gas phase is the balance of

momentum for the porous medium as a whole (2.7), whereby the assumption of slow motions is taken into account.

The balance of momentum for the species water and for the species air in the liquid phase and in the gas phase will be replaced by Fick's laws. These laws correspond to the balance of momentum for the species water and the species air in the liquid phase and in the gas phase and to constitutive assumptions for the stress tensors of the species water and the species air in the liquid phase and in the gas phase and for the exchange of momentum of each of these two species in each of these two phases with the other species in the other phases.

The sum of the balance of momentum of the species water and of the species air in the liquid phase is the balance of momentum of the liquid phase, taking into account the assumption of slow motions. Similarly, the sum of the balance of momentum of the species water and of the species air in the gas phase is the balance of momentum in the gas phase, taking into account the assumption of slow motion.

- **Balance of energy**

Because of the assumption that temperature is everywhere constant, the various balances of energy are not needed.

2.2 Equilibrium restrictions

The processes considered are restricted by certain constraints placed on some variables of the porous medium implied by the assumption of slow motions. The equilibrium conditions considered are: mechanical equilibrium, thermal equilibrium and phase change equilibrium.

- **Mechanical equilibrium**

Due to the assumption of slow motion, inertial terms and terms involving the product of velocities are neglected in the balance of momentum equations.

- **Thermal equilibrium**

We assume that the exchanges of energy between the various species in the various phases are much faster than the evolution of the porous medium, thereby implying that at any space point and time instant their temperatures are equal. Furthermore, we assume that the evolution of the porous medium takes places under isothermal conditions.

- **Phase change equilibrium**

We assume that species water and species air are in equilibrium with respect to phase changes between the liquid phase and the gas phase. In this regard, we will assume that the psychrometric law and Henry's law hold

$$\omega_g^w \rho_g = p_{g0}^w(T) \frac{M_w}{R(273.15+T)} \exp\left(\frac{-(p_g - p_l)M_w}{R(273.15+T)\rho_l} \right) \quad \text{and} \tag{2.8}$$

$$\omega_l^a = (\omega_g^a \rho_g) \frac{R(273.15+T)}{HM_w}, \tag{2.9}$$

where $M_w = 0.018\,kg/mol$ is the molar mass of water, $M_a = 0.02895\,kg/mol$ is the molar mass of air, $R = 8.3144521\times10^{-6}\,MPa\cdot m^3/(°K\cdot mol)$ is the ideal gases constant, T is the temperature in °C, $H = 10^4\,MPa$ is Henry's constant and $p_{g0}^w(T)$ is the water vapour pressure in equilibrium with liquid water through a planar surface, whose expression has been taken to be

$$p_{g0}^w(T) = 136075\exp\left(\frac{-5239.7}{273.15+T}\right), \tag{2.10}$$

where $p_{g0}^w(T)$ is in MPa and T is in °C.

Due to the assumed phase change equilibrium, the exchanges of the species water and of the species air between the liquid phase and the gas phase may be computed by establishing the balances of mass of species water in the liquid phase, the balance of species air in the liquid phase, the balance of species water in the gas phase and the balance of species air in the gas phase

$$\frac{\partial}{\partial t}\{\omega_l^w\rho_l S_l\phi\} + div\{\boldsymbol{i}_l^w + \omega_l^w\rho_l\boldsymbol{q}_l + \omega_l^w S_l\rho_l\phi\boldsymbol{v}_s\} = f_l^w, \tag{2.11}$$

$$\frac{\partial}{\partial t}\{\omega_g^w\rho_g S_g\phi\} + div\{\boldsymbol{i}_g^w + \omega_g^w\rho_g\boldsymbol{q}_g + \omega_g^w S_g\rho_g\phi\boldsymbol{v}_s\} = f_g^w, \tag{2.12}$$

$$\frac{\partial}{\partial t}\{\omega_l^a\rho_l S_l\phi\} + div\{\boldsymbol{i}_l^a + \omega_l^a\rho_l\boldsymbol{q}_l + \omega_l^a S_l\rho_l\phi\boldsymbol{v}_s\} = f_l^a \quad \text{and} \tag{2.13}$$

$$\frac{\partial}{\partial t}\{\omega_g^a\rho_g S_g\phi\} + div\{\boldsymbol{i}_g^a + \omega_g^a\rho_g\boldsymbol{q}_g + \omega_g^a S_g\rho_g\phi\boldsymbol{v}_s\} = f_g^a, \tag{2.14}$$

where f_l^w and f_g^w are the mass rates of species water being converted to the liquid and gas phases (equations (2.11), (2.13) and the balance of the species water (2.2) imply $f_l^w + f_g^w = 0$) and f_l^a and f_g^a are the mass rates of species air being converted to the liquid and gas phases (equations (2.12), (2.14) and the balance of the species air (2.3) imply $f_l^a + f_g^a = 0$).

2.3 Constitutive relations

Constitutive relations characterize the material properties of the components of the considered porous medium as well as their interactions.

- **Constitutive relations for the porous medium**

The water retention curve determines the degree of saturation S_l of the porous medium as a function of the pressure of the liquid phase p_l and the pressure of the gas phase p_g. We have used van Genuchten's water retention curve

$$S_l = \begin{cases} \left(1+\left(\dfrac{p_g - p_l}{P_0}\right)^{\frac{1}{1-\lambda}}\right)^{-\lambda} & p_g > p_l \\ 1 & p_g \le p_l \end{cases}, \tag{2.15}$$

where λ $(0<\lambda<1)$ is a model parameter and P_0 $(P_0>0)$ depends on the intrinsic permeability k according to the expression

$$P_0 = \frac{C}{\sqrt[3]{k}}, \tag{2.16}$$

where C $(C>0)$ is a constant. In order to determine this constant, we use the expression

$$C = P_{0\,matrix}\sqrt[3]{k_{matrix}}, \tag{2.17}$$

where $P_{0\,matrix}$ and k_{matrix} are the values of P_0 and k for the matrix of the material, that is, without cracks (see equation (2.22) below).

We use a generalization of Darcy's law to unsaturated conditions for the motion of the liquid and gas phases with respect to the solid phase.

$$\boldsymbol{q}_l = -\frac{k_{rl}(S_l)k(\varepsilon_{vol})}{\mu_l}\left[grad(p_l) - \rho_l \boldsymbol{g}\right] \quad \text{and} \tag{2.18}$$

$$\boldsymbol{q}_g = -\frac{k_{rg}(S_g)k(\varepsilon_{vol})}{\mu_g}\left[grad(p_g) - \rho_g \boldsymbol{g}\right], \tag{2.19}$$

where the relative permeabilities of the liquid and gas phases are given by

$$k_{rl}(S_l) = S_l^{1/2}\left[1-(1-S_l^{1/\lambda_l})^{\lambda_l}\right]^2 \quad \text{and} \tag{2.20}$$

$$k_{rg}(S_g) = A_g S_g^{n_g}, \tag{2.21}$$

where λ_l, n_g and A_g are model parameters and the intrinsic permeability, as proposed in Olivella et al (2008), is assumed to be isotropic and given by the sum of the contributions from the matrix (uniform flow) and from the cracks (Poiseuille flow depending on the crack aperture)

$$k(\varepsilon_{vol}) = k_{matrix} + \frac{1}{12a}[b(\varepsilon_{vol})]^3 \tag{2.22}$$

where k_{matrix} is the constant matrix permeability, a is the mean spacing between cracks and $b(\varepsilon_{vol})$ denotes the crack aperture as a function of ε_{vol} defined by

$$b(\varepsilon_{vol}) = \begin{cases} b_0 & \varepsilon_{vol} \le \varepsilon_{vol0} \\ b_0 + (\varepsilon_{vol} - \varepsilon_{vol0})a & \varepsilon_{vol0} \le \varepsilon_{vol} \le \varepsilon_{vol0} + (b_{max} - b_0)/a \\ b_{max} & \varepsilon_{vol0} + (b_{max} - b_0)/a \le \varepsilon_{vol} \end{cases} \tag{2.23}$$

where ε_{vol0} is the volumetric deformation at which the crack aperture starts increasing, b_0 is the minimum crack aperture, b_{max} is the maximum crack aperture and a is the mean spacing between cracks.

The dependency of the intrinsic permeability on the crack aperture, which is related to the volumetric deformation of the porous medium, is very important for the modelling of the HG-A test. Figure 2-1 below shows an example of the dependency of the intrinsic permeability (in logarithmic scale) and of the crack aperture with the volumetric deformation, whereby the parameters have values suitable for the HG-A test. Note the large variations in the intrinsic permeability induced by the volumetric deformations.

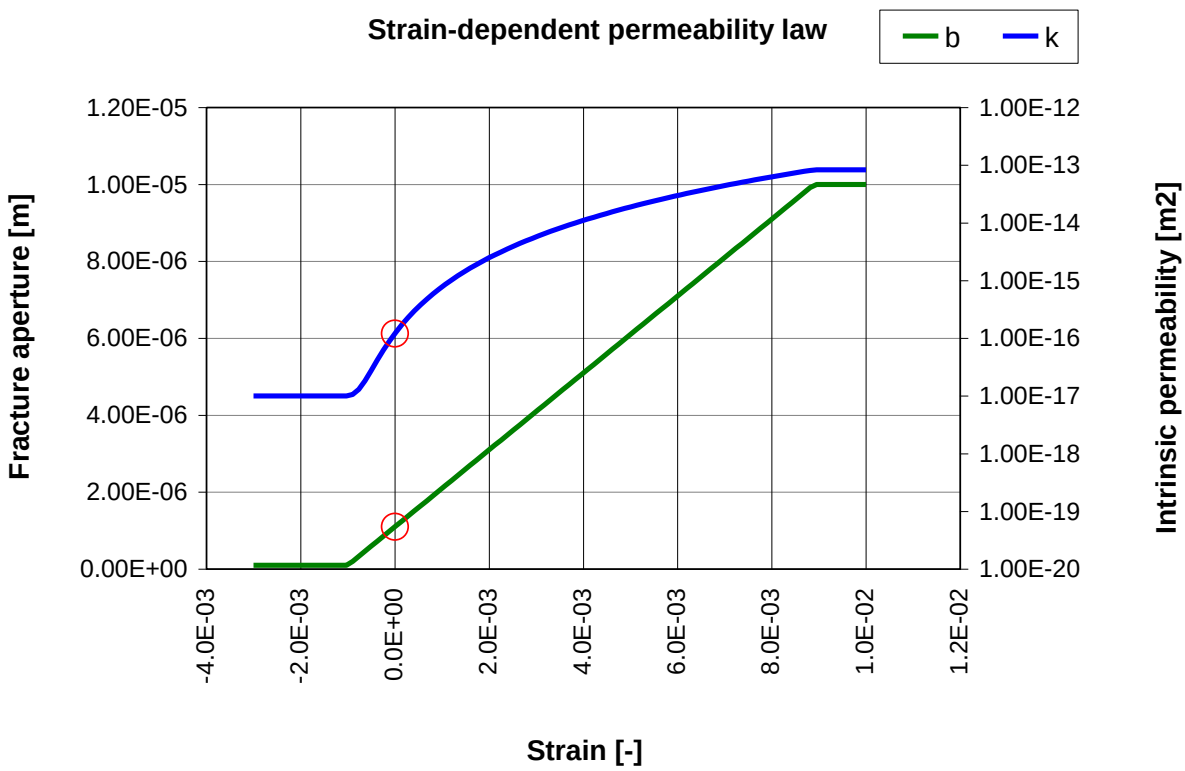

Figure 2-1. Evolution of the crack aperture (left) and of the intrinsic permeability (right, in logarithmic scale) as a function of volumetric deformation ($k_{matrix} = 10^{-17} m^2$, $b_0 = 10^{-7} m$, $b_{\max} = 10^{-5} m$, $a = 10^{-3} m$, $\varepsilon_{vol0} = -10^{-3}$). Circles indicate the initial state, at $\varepsilon = 0$.

Finally, we use Fick's law for the motion of the water and gas species with respect to the liquid and gas phases.

$$\boldsymbol{i}_l^w = -\rho_l(\phi S_l D_l^w \boldsymbol{I} + \boldsymbol{D}_l')grad(\omega_l^w) \quad \text{(negligible term with respect to conduction)} \tag{2.24}$$

$$\boldsymbol{i}_l^a = -\rho_l(\phi S_l D_l^a \boldsymbol{I} + \boldsymbol{D}_l')grad(\omega_l^a) \quad \text{(important term)} \tag{2.25}$$

$$\boldsymbol{i}_g^w = -\rho_g(\phi S_g D_g^w \boldsymbol{I} + \boldsymbol{D}_g')grad(\omega_g^w) \tag{2.26}$$

$$\boldsymbol{i}_g^a = -\rho_g(\phi S_g D_g^a \boldsymbol{I} + \boldsymbol{D}_g')grad(\omega_g^a)\,, \tag{2.27}$$

where the diffusion coefficients of the water species in the liquid and gas phases are given by

$$D_l^w = D_l^a = D_l \exp\left(\frac{-Q_l}{R(273.15+T)}\right) \tag{2.28}$$

$$D_g^w = D_g^a = D_g \frac{(273.15+T)^{Q_g}}{p_g} \tag{2.29}$$

where D_l and Q_l (resp. D_g and Q_g) are model parameters for the liquid (resp. gas) phase and the dispersion tensors are given by

$$\boldsymbol{D}_l' = d_l^t\,|\boldsymbol{q}_l|\,\boldsymbol{I} + (d_l^l - d_l^t)\frac{1}{|\boldsymbol{q}_l|}\boldsymbol{q}_l \otimes \boldsymbol{q}_l \qquad \text{(in 1D flow, } \boldsymbol{D}_l'\boldsymbol{e}_z = d_l^l\,|\boldsymbol{q}_l|\,\boldsymbol{e}_z\text{)} \tag{2.30}$$

$$\boldsymbol{D}_g' = d_g^t\,|\boldsymbol{q}_g|\,\boldsymbol{I} + (d_g^l - d_g^t)\frac{1}{|\boldsymbol{q}_g|}\boldsymbol{q}_g \otimes \boldsymbol{q}_g\,, \qquad \text{(in 1D flow, } \boldsymbol{D}_g'\boldsymbol{e}_z = d_g^l\,|\boldsymbol{q}_g|\,\boldsymbol{e}_z\text{)} \tag{2.31}$$

where d_l^l and d_l^t (resp. d_g^l and d_g^t) are the longitudinal and transversal dispersivity coefficients of the liquid (resp. gas) phase.

- **Constitutive relations for the solid phase**

It has been assumed that the solid phase has only one species. The solid phase has been assumed to be incompressible

$$\rho_s = const\,. \tag{2.32}$$

- **Constitutive relations for the liquid phase**

The liquid phase has also been assumed to be a mixture of two species (water and dissolved air). However, their properties have been assumed to be independent of the amount of dissolved air. It has been assumed that the liquid phase is incompressible.

$$\rho_l(T) = 1002.6\exp(-0.00024T) \tag{2.33}$$

$$\mu_l(T) = 2.1\times10^{-12}\exp(1808.5/(273.15+T)) \tag{2.34}$$

where ρ_l is in kg/m^3, $\mu_l(T)$ is in MPa·s and T is in °C.

- **Constitutive relations for the gas phase**

The gas phase has been assumed to be a mixture of two species (water vapour and air). It has been assumed that each species (water vapour and dry air) is an ideal gas and that Dalton's law holds.

$$\mu_g(T) = 1.48\times10^{-12}\sqrt{(273.15+T)/(1+119/(273.15+T))} \tag{2.35}$$

$$p_g = p_g^a + p_g^w \qquad \text{Dalton's law} \tag{2.36}$$

$$p_g^a = \omega_g^a \rho_g \frac{R(273.15+T)}{M_a} \qquad \text{ideal gas law for dry air} \tag{2.37}$$

$$p_g^w = \omega_g^w \rho_g \frac{R(273.15+T)}{M_w} \qquad \text{ideal gas law for water vapour} \tag{2.38}$$

where $\mu_g(T)$ is in MPa·s and T is in °C.

- **Constitutive relations for the solid skeleton**

Finally, we will also assume that a generalized form of Terzaghi's effective stress principle holds and the deformation stress tensor of the solid phase $\boldsymbol{\varepsilon}$ depends on the total stress tensor $\boldsymbol{\sigma}$, the liquid phase pressure p_l and the gas phase pressure p_g through the effective stress tensor $\boldsymbol{\sigma}'$ defined (using the continuum mechanics sign convention) by the expression.

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} + p^* \boldsymbol{I}, \tag{2.39}$$

where the effective fluid pressure is defined by

$$p^* = \max(p_l, p_g). \tag{2.40}$$

The solid skeleton will be assumed to be an isotropic linear poroelastic material obeying to Hooke's law. In a Cartesian coordinate system oriented along the common principal directions of the stress and strain tensors, Hooke's law reads

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{E'}(\Delta\sigma_1' - \nu'\Delta\sigma_2' - \nu'\Delta\sigma_3') \\
\varepsilon_2 &= \frac{1}{E'}(-\nu'\Delta\sigma_1' + \Delta\sigma_2' - \nu'\Delta\sigma_3') \\
\varepsilon_3 &= \frac{1}{E'}(-\nu'\Delta\sigma_1' - \nu'\Delta\sigma_2' + \Delta\sigma_3'),
\end{aligned} \tag{2.41}$$

where E' is the effective Young's modulus and ν' is the effective Poisson's ratio. We will assume that $-1 < \nu' < 1/2$, so that Hooke's law (2.41) is invertible.

2.4 Field equations

Combining the balance equations, the equilibrium restrictions and the constitutive relations we get a system of differential equations with the displacement vector of the solid skeleton $\boldsymbol{u}_s$, the pressure of the liquid phase p_l and the pressure of the gas phase p_g as unknown functions of the space position and time.

The balance of mass of the solid phase (using $\rho_s = const$) and the balance of water mass read

$$\frac{\partial \phi}{\partial t} + div(\phi \boldsymbol{v}_s) = div(\boldsymbol{v}_s) \tag{2.42}$$

$$\begin{aligned}\phi \frac{\partial}{\partial t}(\omega_l^w \rho_l S_l + \omega_g^w \rho_g S_g) + (\omega_l^w \rho_l S_l + \omega_g^w \rho_g S_g)\frac{\partial \phi}{\partial t} + div(\boldsymbol{i}_l^w + \omega_l^w \rho_l \boldsymbol{q}_l + \boldsymbol{i}_g^w + \omega_g^w \rho_g \boldsymbol{q}_g) \\ + (\omega_g^w S_g \rho_g + \omega_l^w S_l \rho_l) div(\phi \boldsymbol{v}_s) + (\phi \boldsymbol{v}_s) grad(\omega_g^w S_g \rho_g + \omega_l^w S_l \rho_l) = 0\end{aligned} \tag{2.43}$$

Combining these equations, we get

$$\begin{aligned}\phi \left\{ \frac{\partial}{\partial t}(\omega_l^w \rho_l S_l + \omega_g^w \rho_g S_g) + \boldsymbol{v}_s grad(\omega_l^w \rho_l S_l + \omega_g^w \rho_g S_g) \right\} + \\ + (\omega_l^w \rho_l S_l + \omega_g^w \rho_g S_g) div(\boldsymbol{v}_s) + div(\boldsymbol{i}_l^w + \omega_l^w \rho_l \boldsymbol{q}_l + \boldsymbol{i}_g^w + \omega_g^w \rho_g \boldsymbol{q}_g) = 0\end{aligned} \tag{2.44}$$

Neglecting the second term and using the relation (small strains)

$$div(\boldsymbol{v}_s) = \frac{\partial \varepsilon_{s,vol}}{\partial t} \tag{2.45}$$

we arrive to

$$\begin{aligned}\phi \frac{\partial}{\partial t}(\omega_l^w \rho_l S_l + \omega_g^w \rho_g S_g) + (\omega_l^w \rho_l S_l + \omega_g^w \rho_g S_g)\frac{\partial \varepsilon_{s,vol}}{\partial t} \\ + div(\boldsymbol{i}_l^w + \omega_l^w \rho_l \boldsymbol{q}_l + \boldsymbol{i}_g^w + \omega_g^w \rho_g \boldsymbol{q}_g) = 0\end{aligned} \tag{2.46}$$

Similarly, the balance of mass of the solid phase (using $\rho_s = const$) and the balance of air mass, yield

$$\begin{aligned}\phi \frac{\partial}{\partial t}(\omega_l^a \rho_l S_l + \omega_g^a \rho_g S_g) + (\omega_l^a \rho_l S_l + \omega_g^a \rho_g S_g)\frac{\partial \varepsilon_{s,vol}}{\partial t} \\ + div(\boldsymbol{i}_l^a + \omega_l^a \rho_l \boldsymbol{q}_l + \boldsymbol{i}_g^a + \omega_g^a \rho_g \boldsymbol{q}_g) = 0\end{aligned} \tag{2.47}$$

On the other hand, using $\rho_s = const$ in the integrated form (2.6) of the balance of mass of the solid phase, we arrive to

$$1 - \phi = \frac{1 - \phi^0}{1 + \varepsilon_{s,vol}}. \tag{2.48}$$

This relation allows to express the porosity ϕ as a function of the derivatives of the displacement vector of the solid phase $\boldsymbol{u}_s$.

Therefore, upon using the rest of the balance equations, the equilibrium restrictions and the constitutive relations, the balance of mass of the species water, the balance of mass of the species air and the balance of momentum of the porous medium

$$\phi\frac{\partial}{\partial t}(\omega_l^w\rho_l S_l+\omega_g^w\rho_g S_g)+(\omega_l^w\rho_l S_l+\omega_g^w\rho_g S_g)\frac{\partial\varepsilon_{s,vol}}{\partial t} \\ +div(\boldsymbol{i}_l^w+\omega_l^w\rho_l\boldsymbol{q}_l+\boldsymbol{i}_g^w+\omega_g^w\rho_g\boldsymbol{q}_g)=0 \tag{2.49}$$

$$\phi\frac{\partial}{\partial t}(\omega_l^a\rho_l S_l+\omega_g^a\rho_g S_g)+(\omega_l^a\rho_l S_l+\omega_g^a\rho_g S_g)\frac{\partial\varepsilon_{s,vol}}{\partial t} \\ +div(\boldsymbol{i}_l^a+\omega_l^a\rho_l\boldsymbol{q}_l+\boldsymbol{i}_g^a+\omega_g^a\rho_g\boldsymbol{q}_g)=0 \tag{2.50}$$

$$div(\boldsymbol{\sigma}'-p^*\boldsymbol{I})+\rho\boldsymbol{g}=\boldsymbol{0}, \tag{2.51}$$

may be written in terms of the displacement vector of the solid skeleton $\boldsymbol{u}_s$, the pressure of the liquid phase p_l and the pressure of the gas phase p_g and their partial derivatives. Appendix A1 shows the expressions of some of the functions appearing in these equations as functions of the liquid phase p_l and on the pressure of gas phase p_g and their first partial derivatives.

This system of differential equations in the unknown functions $\boldsymbol{u}_s$, p_l and p_g will be referred to as the system of field equations. In order to solve this system of differential equations, suitable initial and boundary conditions reflecting the conditions of the HG-A test must be provided. Once the unknown functions are determined as a function of the space point and of time, all the functions of interest are found from them

2.5 Transition between saturated and unsaturated states

As already mentioned, the unknown functions $\boldsymbol{u}_s$, p_l and p_g are defined at every point of the geometrical domain and at all times and are found by solving the system of field equations with suitable initial and boundary conditions.

On the other hand, using the psychrometric law and Henry's law as equilibrium restrictions (stating the phase change equilibrium of water species and of air species between a gas phase and a liquid phase made of these two species), the ideal gases law for the water vapor, the ideal gases law for dry air and Dalton's law, it is possible to express p_g, ρ_g, ω_g^w and ω_g^a as a function of p_l, ρ_l, ω_l^w and ω_l^a or, conversely, to express p_l, ρ_l, ω_l^w and ω_l^a as a function of p_g, ρ_g, ω_g^w and ω_g^a.

We note that the water retention curve is a continuous function $S_l=f(p_g-p_l)$ such that $S_l=1$ (saturated state) if and only if $p_g\le p_l$ and $S_l<1$ if and only if $p_g>p_l$. This has the following consequences

- **Mechanical.** The effective fluid pressure $p^*=f(p_l,p_g)$ is a continuous function of p_l and p_g, such that $p^*=p_l$ if and only if $p_g\le p_l$ and $p^*=p_g$ if and only if $p_g\ge p_l$. Consequently, the effective stress tensor $\boldsymbol{\sigma}'$ is Terzaghi's effective stress tensor $\boldsymbol{\sigma}+p_l\boldsymbol{I}$ if and only if $p_g\le p_l$ and is the net stress tensor $\boldsymbol{\sigma}+p_g\boldsymbol{I}$ if and only if $p_g\ge p_l$.

- **Hydraulic.** The relative permeability of the gas phase is a continuous function $k_{rg} = f(1-S_l)$ of S_l such that $k_{rg} = 0$ if and only if $S_l = 1$. If $S_l = 1$, then Darcy's law for the gas phase yields $\boldsymbol{q}_g = \boldsymbol{0}$. The mechanical dispersion tensor of the gas phase is a continuous function $\boldsymbol{D}'_g = f\left(\boldsymbol{q}_g\right)$ such that $\boldsymbol{D}'_g = \boldsymbol{0}$ when $\boldsymbol{q}_g = \boldsymbol{0}$. Finally, if $\boldsymbol{q}_g = \boldsymbol{0}$ then Fick's law for the water species and for the air species in the gas phase yield $\boldsymbol{i}_g^w = \boldsymbol{0}$ and $\boldsymbol{i}_g^a = \boldsymbol{0}$. Consequently, if $p_g \le p_l$ then $S_g = 0$ and $\boldsymbol{q}_g = \boldsymbol{i}_g^w = \boldsymbol{i}_g^a = \boldsymbol{0}$. In this case, all terms pertaining to the gas phase disappear from the balance of mass equations.

The water retention curve determines whether the porous medium is saturated or unsaturated, but in the field equations some terms vanish if the porous medium is saturated. More precisely

- **Saturated porous medium.** If $p_g \le p_l$, then the porous medium saturates ($S_l = 1$) and the terms $\boldsymbol{q}_g$, $\boldsymbol{i}_g^w$ and $\boldsymbol{i}_g^a$ disappear from the field equations, which reduce to the field equations of a saturated porous medium (the liquid phase has the species water and the species air) with $\boldsymbol{\sigma}' = \boldsymbol{\sigma} + p_l \boldsymbol{I}$ being Terzaghi's effective stress

$$\phi \frac{\partial}{\partial t}(\omega_l^w \rho_l) + (\omega_l^w \rho_l)\frac{\partial \varepsilon_{s,vol}}{\partial t} + div(\boldsymbol{i}_l^w + \omega_l^w \rho_l \boldsymbol{q}_l) = 0 \tag{2.52}$$

$$\phi \frac{\partial}{\partial t}(\omega_l^a \rho_l) + (\omega_l^a \rho_l S_l)\frac{\partial \varepsilon_{s,vol}}{\partial t} + div(\boldsymbol{i}_l^a + \omega_l^a \rho_l \boldsymbol{q}_l) = 0 \tag{2.53}$$

$$div(\boldsymbol{\sigma}' - p_l \boldsymbol{I}) + \rho \boldsymbol{g} = \boldsymbol{0} \tag{2.54}$$

Using (2.11) and (2.13) in the two first equations, it follows that $f_l^w = 0$ and $f_l^a = 0$. That is, there are no mass contributions from the non-existent gas phase. As explained earlier, the pressure of the gas phase p_g at a point of the porous medium and at a time instant is the pressure of a gas (made of species water and species air) which is in equilibrium (with respect to exchanges of species water and of species air) with the liquid phase (made of species water and species air) at the considered point of the porous medium and at the considered time instant.

- **Unsaturated porous medium.** If $p_g > p_l$, then the porous medium desaturates ($S_l < 1$), the terms $\boldsymbol{q}_g$, $\boldsymbol{i}_g^w$ and $\boldsymbol{i}_g^a$ appear in the field equations, which are the field equations of an unsaturated porous medium (the liquid phase and the gas phase have the species water and the species air) with $\boldsymbol{\sigma}' = \boldsymbol{\sigma} + p_g \boldsymbol{I}$ being the net stress

$$\begin{aligned}\phi \frac{\partial}{\partial t}(\omega_l^w \rho_l S_l + \omega_g^w \rho_g S_g) + (\omega_l^w \rho_l S_l + \omega_g^w \rho_g S_g)\frac{\partial \varepsilon_{s,vol}}{\partial t} \\ + div(\boldsymbol{i}_l^w + \omega_l^w \rho_l \boldsymbol{q}_l + \boldsymbol{i}_g^w + \omega_g^w \rho_g \boldsymbol{q}_g) = 0\end{aligned} \tag{2.55}$$

$$\begin{aligned}\phi \frac{\partial}{\partial t}(\omega_l^a \rho_l S_l + \omega_g^a \rho_g S_g) + (\omega_l^a \rho_l S_l + \omega_g^a \rho_g S_g)\frac{\partial \varepsilon_{s,vol}}{\partial t} \\ + div(\boldsymbol{i}_l^a + \omega_l^a \rho_l \boldsymbol{q}_l + \boldsymbol{i}_g^a + \omega_g^a \rho_g \boldsymbol{q}_g) = 0\end{aligned} \tag{2.56}$$

$$div(\boldsymbol{\sigma}' - p_g \boldsymbol{I}) + \rho \boldsymbol{g} = \boldsymbol{0}\,. \tag{2.57}$$

3 1D Modelling

By use of certain simplifying assumptions, it is possible to reduce the problem to a coupled system of two differential equations in two unknown functions of a geometric variable z defined on a 1D domain, representing the part of the EDZ in contact with the megapacker, and of time t. The boundary conditions at the end of this 1D domain close to the test section are derived from the protocols (time evolutions) of the volume injection rate of the liquid phase and of the volume injection rate of the gas phase. The boundary conditions at the end of this domain close to the liner are generalized conditions involving the mass rates of the liquid phase and of the gas phase and the values of the external pressures of the liquid phase and of the gas phase. The protocol of the megapacker pressure is transformed into a body term on the 1D domain. The initial conditions for the unknown functions are derived from the assumed conditions of the test. This coupled system of equations is solved using the FEM. Once the two unknown functions of the geometric 1D variable and of time are known, all the field variables both on the Opalinus clay and on the EDZ are determined in closed form using those two unknown functions.

3.1 Geometry and coordinate system

Figure 3-1 below shows the assumed geometry of the portion of the EDZ and Opalinus clay surrounding the HG-A tunnel comprised between the two planes orthogonal to the tunnel axis that pass through each of the ends of the megapacker.

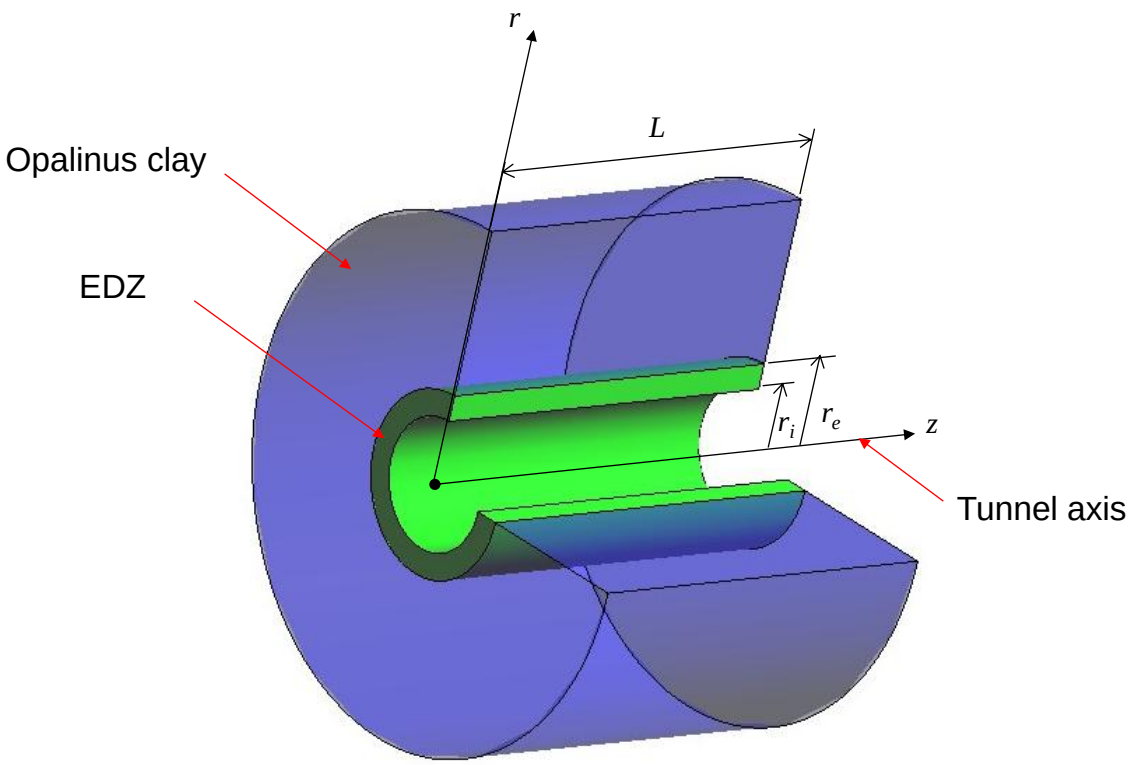

Figure 3-1. Geometry and coordinate system around the HG-A tunnel.

A cylindrical coordinate system (r, θ, z) has been defined such that (1) its origin is at the intersection of the tunnel axis with the plane orthogonal to it that passes through the

megapacker end close to the test section; (2) the z axis is parallel to the tunnel axis and is oriented towards the tunnel mouth; (3) the angle θ is measured with respect to the horizontal. The origin of time has been set at 15.04.2006.

In that figure, the definitions of the geometric parameters that will be used are also shown, namely: L is the length of the megapacker ($L = 3.00\,m$); r_i is the radius of the tunnel ($r_i = 0.50\,m$); and r_e is the radius of the interface between the EDZ and the Opalinus clay ($r_e = 0.70\,m$). The considered geometry is symmetric around the tunnel axis, hence the EDZ has a uniform thickness $r_e - r_i$.

3.2 Simplifying assumptions

The considered simplifying assumptions are of mechanical and hydraulic nature.

Mechanical. Both on the EDZ and on the Opalinus clay, it will be assumed that (1) there is axisymmetry about the tunnel axis (z axis); (2) there are no volume forces (gravity is neglected); and (3) slices $z = const$ move independently and in plane strain.

Hydraulic. Both on the EDZ and on the Opalinus clay, it will be assumed that (1) there is axisymmetry about the tunnel axis (z axis); and (2) there are no volume forces (gravity is neglected). Furthermore, on the EDZ, it will be assumed that (3)^{EDZ} the flows of the liquid phase and of the gas phase are parallel to the tunnel axis (z axis); and, on the Opalinus clay, it will be assumed that (3)^{OPA} there is no gas phase (saturated state), there is no flow of the liquid phase (undrained conditions) and there is no air in the liquid phase.

3.3 Consequences of the simplifying assumptions

Mechanical axisymmetry about the tunnel axis (z axis) requires that the displacement fields on the EDZ and on the Opalinus clay do not depend on θ, that is

$$\begin{aligned} \boldsymbol{u}^{EDZ} &= u_r^{EDZ}(r,z,t)\boldsymbol{e}_r \qquad (r_i \le r \le r_e,\ 0 \le z \le L), \\ \boldsymbol{u}^{OPA} &= u_r^{OPA}(r,z,t)\boldsymbol{e}_r \qquad (r_e \le r,\ 0 \le z \le L). \end{aligned} \tag{3.1}$$

Independent motion and in plane strain of slices $z = const$ amounts to neglect the shear components of the deformation tensor field on the EDZ and on the Opalinus clay implied by the displacement fields on the EDZ and on the Opalinus clay, that is

$$\begin{aligned} \boldsymbol{\varepsilon}^{EDZ} &= \frac{\partial u_r^{EDZ}}{\partial r}(r,z,t)\boldsymbol{e}_r \otimes \boldsymbol{e}_r + \frac{u_r^{EDZ}(r,z,t)}{r}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta \qquad (r_i \le r \le r_e,\ 0 \le z \le L), \\ \boldsymbol{\varepsilon}^{OPA} &= \frac{\partial u_r^{OPA}}{\partial r}(r,z,t)\boldsymbol{e}_r \otimes \boldsymbol{e}_r + \frac{u_r^{OPA}(r,z,t)}{r}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta \qquad (r_e \le r,\ 0 \le z \le L). \end{aligned} \tag{3.2}$$

Hydraulic axisymmetry about the tunnel axis (z axis) requires that, on the EDZ and on the Opalinus clay, the liquid phase pressure p_l does not depend on θ and, on the EDZ, the gas phase pressure p_g does not depend on θ. The assumption that, on the EDZ, the flow of the liquid phase $\boldsymbol{q}_l$ and the flow of the gas phase $\boldsymbol{q}_g$ are parallel to the tunnel axis, and Darcy's

laws (2.18) and (2.19) imply that the liquid phase pressure p_l and the gas phase pressure p_g do not depend on r (nor on θ). Summarizing,

$$\begin{aligned} p_l^{EDZ} &= p_l^{EDZ}(z,t) \\ p_g^{EDZ} &= p_g^{EDZ}(z,t) \\ p_l^{OPA} &= p_l^{OPA}(r,z,t) \end{aligned} \tag{3.3}$$

Using this result and the expression of the retention curve (2.15), we conclude that on the EDZ S_l and S_g do not depend on r nor on θ.

$$\begin{aligned} S_l^{EDZ} &= S_l^{EDZ}(z,t) \\ S_g^{EDZ} &= S_g^{EDZ}(z,t) \end{aligned} \tag{3.4}$$

Since, on the EDZ, the liquid phase pressure p_l and the gas phase pressure p_g do not depend on r nor on θ, the definition (2.40) of the effective pressure $p*$ shows that it does not depend on r nor on θ,

$$p*^{EDZ} = p*^{EDZ}(z,t). \tag{3.5}$$

Therefore, on any slice $z = const$ and any time t, we can use the results of Appendix A4 showing the results of the integration of the balance of momentum for the porous medium, namely the expressions for the non-vanishing components of the displacement vector, strain tensor and total stress tensor on the EDZ

$$\begin{aligned} u_r^{EDZ}(r,z,t) &= A^{EDZ}(z,t)r + \frac{B^{EDZ}(z,t)}{r} \\ \varepsilon_r^{EDZ}(r,z,t) &= A^{EDZ}(z,t) - \frac{B^{EDZ}(z,t)}{r^2} \\ \varepsilon_\theta^{EDZ}(r,z,t) &= A^{EDZ}(z,t) + \frac{B^{EDZ}(z,t)}{r^2} \\ \Delta\sigma_r^{EDZ}(r,z,t) &= \frac{E'^{EDZ}}{(1+\nu'^{EDZ})(1-2\nu'^{EDZ})} A^{EDZ}(z,t) - \frac{E'^{EDZ}}{(1+\nu'^{EDZ})} \frac{B^{EDZ}(z,t)}{r^2} - \Delta p*(z,t), \\ \Delta\sigma_\theta^{EDZ}(r,z,t) &= \frac{E'^{EDZ}}{(1+\nu'^{EDZ})(1-2\nu'^{EDZ})} A^{EDZ}(z,t) + \frac{E'^{EDZ}}{(1+\nu'^{EDZ})} \frac{B^{EDZ}(z,t)}{r^2} - \Delta p*(z,t), \\ \Delta\sigma_z^{EDZ}(r,z,t) &= \frac{E'^{EDZ}}{(1+\nu'^{EDZ})(1-2\nu'^{EDZ})} [2\nu'^{EDZ} A^{EDZ}(z,t)] - \Delta p*(z,t), \end{aligned} \tag{3.6}$$

and the non-vanishing components of the displacement vector, strain tensor, total stress tensor and pore water pressure on the Opalius clay

$$
\begin{aligned}
&u_r^{OPA}(r,z,t) = \frac{B^{OPA}(z,t)}{r}, \\
&u_r^{OPA}(r,z,t) = \frac{B^{OPA}(z,t)}{r}, \\
&\varepsilon_r^{OPA}(r,z,t) = -\frac{B^{OPA}(z,t)}{r^2}, \\
&\varepsilon_\theta^{OPA}(r,z,t) = \frac{B^{OPA}(z,t)}{r^2}, \\
&\Delta\sigma_r^{OPA}(r,z,t) = -\frac{E'^{OPA}}{(1+\nu'^{OPA})}\frac{B^{OPA}(z,t)}{r^2}, \\
&\Delta\sigma_\theta^{OPA}(r,z,t) = \frac{E'^{OPA}}{(1+\nu'^{OPA})}\frac{B^{OPA}(z,t)}{r^2}, \\
&\Delta\sigma_z^{OPA}(r,z,t) = 0 \quad \text{and} \\
&\Delta p*^{OPA}(r,z,t) = 0.
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
&A^{EDZ}(z,t) = A_{\Delta\sigma}^{EDZ}\Delta\sigma(z,t) + A_{\Delta p*}^{EDZ}\Delta p*^{EDZ}(z,t), \\
&B^{EDZ}(z,t) = B_{\Delta\sigma}^{EDZ}\Delta\sigma(z,t) + B_{\Delta p*}^{EDZ}\Delta p*^{EDZ}(z,t) \quad \text{and} \\
&B^{OPA}(z,t) = B_{\Delta\sigma}^{OPA}\Delta\sigma(z,t) + B_{\Delta p*}^{OPA}\Delta p*^{EDZ}(z,t),
\end{aligned}
\tag{3.8}
$$

Here $\Delta\sigma(z,t) = \sigma(z,t) - \sigma(z,0)$ is the variation with respect to its initial value of the total pressure applied to the tunnel wall at slice z and time t (in the HG-A test, it does not depend on z) and the expressions for the 6 constants $A_{\Delta\sigma}^{EDZ}$, $A_{\Delta p*}^{EDZ}$, $B_{\Delta\sigma}^{EDZ}$, $B_{\Delta p*}^{EDZ}$, $B_{\Delta\sigma}^{OPA}$ and $B_{\Delta p*}^{OPA}$ are given in Appendix A4 as functions of the elastic properties of the EDZ (E'^{EDZ} and ν'^{EDZ}), the elastic properties of the Opalinus clay (E'^{OPA} and ν'^{OPA}) and of the geometry of the test (r_i and r_e).

From these relations, it follows that the volumetric deformation on the EDZ and the increment of the effective pressure (equal to the liquid phase pressure p_l) on the Opalinus clay do not depend on r nor on θ:

$$\varepsilon_{vol}^{EDZ}(z,t) = \varepsilon_r^{EDZ}(r,z,t) + \varepsilon_\theta^{EDZ}(r,z,t) = 2[A_{\Delta\sigma}^{EDZ}\Delta\sigma(z,t) + A_{\Delta p*}^{EDZ}\Delta p*^{EDZ}(z,t)] \tag{3.9}$$

$$\Delta p*^{OPA}(z,t) = -C^{OPA}(z,t) \tag{3.10}$$

Using the expression (2.20) for the relative permeability to the liquid phase, the expression (2.21) for the relative permeability of the gas phase, the expression (2.22) for the intrinsic permeability and results (3.3), (3.4) and (3.9), the expression (2.18) for Darcy's law for the liquid phase and the expression (2.19) for Darcy's law for the gas phase read

$$\boldsymbol{q}_l^{EDZ}(z,t) = -\frac{k_{rl}^{EDZ}(S_l^{EDZ}(z,t))k^{EDZ}(\varepsilon_{vol}^{EDZ}(z,t))}{\mu_l}\frac{\partial p_l^{EDZ}}{\partial z}(z,t)\boldsymbol{e_z} \quad \text{and} \tag{3.11}$$

$$\boldsymbol{q}_g^{EDZ}(z,t) = -\frac{k_{rg}^{EDZ}(S_g^{EDZ}(z,t))k^{EDZ}(\varepsilon_{vol}^{EDZ}(z,t))}{\mu_g}\frac{\partial p_g^{EDZ}}{\partial z}(z,t)\boldsymbol{e_z}, \tag{3.12}$$

showing that, on the EDZ, the liquid phase flow $\boldsymbol{q}_l$ and the gas phase low $\boldsymbol{q}_g$ do not depend on r nor on θ.

Since, on the EDZ, the liquid phase pressure and the gas phase pressure do not depend on r nor on θ, all variables depending only on them will not depend on r nor on θ. In particular,

$$\begin{aligned}
&\omega_l^{wEDZ} = \omega_l^{wEDZ}(z,t),\\
&\omega_l^{aEDZ} = \omega_l^{aEDZ}(z,t),\\
&\omega_g^{wEDZ} = \omega_g^{wEDZ}(z,t),\\
&\omega_g^{aEDZ} = \omega_g^{aEDZ}(z,t) \quad \text{and}\\
&\rho_g^{EDZ} = \rho_g^{EDZ}(z,t).
\end{aligned} \tag{3.13}$$

Since the volumetric deformation ε_{vol}^{EDZ} does not depend on r nor on θ, if the initial porosity ϕ^{0EDZ} does not depend on r nor on θ, from equation (2.48) it follows that the porosity ϕ^{EDZ} does not depend on r nor on θ,

$$\phi^{EDZ} = \phi^{EDZ}(z,t). \tag{3.14}$$

Taking into account Darcy's laws (3.11) and (3.12) and the results (3.13) and (3.14), Fick's equations (2.24) to (2.31) read

$$\boldsymbol{i}_l^{wEDZ}(z,t) = -\rho_l^{EDZ}(z,t)(\phi^{EDZ}(z,t)S_l^{EDZ}(z,t)D_l^w + d_l^l\,|\boldsymbol{q}_l^{EDZ}(z,t)|)\frac{\partial \omega_l^{wEDZ}}{\partial z}(z,t)\boldsymbol{e_z}, \tag{3.15}$$

$$\boldsymbol{i}_l^{aEDZ}(z,t) = -\rho_l^{EDZ}(z,t)(\phi^{EDZ}(z,t)S_l^{EDZ}(z,t)D_l^a + d_l^l\,|\boldsymbol{q}_l^{EDZ}(z,t)|)\frac{\partial \omega_l^{aEDZ}}{\partial z}(z,t)\boldsymbol{e_z}, \tag{3.16}$$

$$\boldsymbol{i}_g^{wEDZ}(z,t) = -\rho_g^{EDZ}(z,t)(\phi^{EDZ}(z,t)S_g^{EDZ}(z,t)D_g^w + d_g^l\,|\boldsymbol{q}_g^{EDZ}(z,t)|)\frac{\partial \omega_g^{wEDZ}}{\partial z}(z,t)\boldsymbol{e_z}, \tag{3.17}$$

$$\boldsymbol{i}_g^{aEDZ}(z,t) = -\rho_g^{EDZ}(z,t)(\phi^{EDZ}(z,t)S_g^{EDZ}(z,t)D_g^a + d_g^l\,|\boldsymbol{q}_g^{EDZ}(z,t)|)\frac{\partial \omega_g^{aEDZ}}{\partial z}(z,t)\boldsymbol{e_z} \quad \text{and,} \tag{3.18}$$

showing that, on the EDZ, the water flow $\boldsymbol{i}_l^w$ and the air flow $\boldsymbol{i}_l^a$ on the liquid phase and the water flow $\boldsymbol{i}_g^w$ and the air flow $\boldsymbol{i}_g^a$ on the gas phase are parallel to the tunnel axis (z axis) and do not depend on r nor on θ.

3.4 Field equations, initial and boundary conditions

Since the balance of momentum has already been integrated in terms of the pressure of the liquid phase and the pressure of the gas phase, the field equations reduce to the balance of

mass for the species water and the balance of mass for the species air. We will use the results obtained from the simplifying assumptions in order to write the field equations, and we will address the initial and boundary conditions considered.

- **Field equations**. Using the relations obtained from the simplifying assumption into the field equations (2.49) and (2.50), we get

$$\begin{aligned}
&\phi\frac{\partial}{\partial t}(\omega_l^w\rho_l S_l+\omega_g^w\rho_g S_g)+(\omega_l^w\rho_l S_l+\omega_g^w\rho_g S_g)\frac{\partial\varepsilon_{s,vol}}{\partial t}\\
&-\frac{\partial}{\partial z}\left\{\rho_l\left(\phi S_l D_l^w+d_l^l\frac{k_{rl}k}{\mu_l}\left|\frac{\partial p_l}{\partial z}\right|\right)\frac{\partial\omega_l^w}{\partial z}+\omega_l^w\rho_l\frac{k_{rl}k}{\mu_l}\frac{\partial p_l}{\partial z}\right\} \qquad (0\le z\le L,\ t\ge 0)\\
&-\frac{\partial}{\partial z}\left\{\rho_g\left(\phi S_g D_g^w+d_g^l\frac{k_{rg}k}{\mu_g}\left|\frac{\partial p_g}{\partial z}\right|\right)\frac{\partial\omega_l^w}{\partial z}+\omega_g^w\rho_g\frac{k_{rg}k}{\mu_g}\frac{\partial p_g}{\partial z}\right\}=0
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
&\phi\frac{\partial}{\partial t}(\omega_l^a\rho_l S_l+\omega_g^a\rho_g S_g)+(\omega_l^a\rho_l S_l+\omega_g^a\rho_g S_g)\frac{\partial\varepsilon_{s,vol}}{\partial t}\\
&-\frac{\partial}{\partial z}\left\{\rho_l\left(\phi S_l D_l^a+d_l^l\frac{k_{rl}k}{\mu_l}\left|\frac{\partial p_l}{\partial z}\right|\right)\frac{\partial\omega_l^a}{\partial z}+\omega_l^a\rho_l\frac{k_{rl}k}{\mu_l}\frac{\partial p_l}{\partial z}\right\} \qquad (0\le z\le L,\ t\ge 0)\\
&-\frac{\partial}{\partial z}\left\{\rho_g\left(\phi S_g D_g^a+d_g^l\frac{k_{rg}k}{\mu_g}\left|\frac{\partial p_g}{\partial z}\right|\right)\frac{\partial\omega_g^a}{\partial z}+\omega_g^a\rho_g\frac{k_{rg}k}{\mu_g}\frac{\partial p_g}{\partial z}\right\}=0
\end{aligned} \tag{3.20}$$

Where, using the chain rule, all partial derivatives with respect to t and z are written in terms of the partial derivatives of p_l and p_g with respect to t and z.

Therefore, equations (3.19) and (3.20) form a (coupled) system of two second order partial differential equations in the unknown fields $p_l(z,t)$ and $p_g(z,t)$. In order to single out the solution to the problem, we impose the following initial and boundary conditions.

- **Initial conditions**. They are the prescription of the distributions of the pressures of the liquid and gas phases.

$$\begin{aligned}
&p_l(z,0)=(p_l)_0(z) && (0\le z\le L)\\
&p_g(z,0)=(p_g)_0(z) && (0\le z\le L)
\end{aligned} \tag{3.21}$$

Recall that, in the present approach, it is assumed that p_l and p_g do not depend on r nor on θ, so that their initial values can only depend on z. We will use $(p_l)_0(z)=p^{atm}$ and $(p_g)_0(z)=p^{atm}$, where p^{atm} is the atmospheric pressure at the test site, so that, according to the water retention curve, the EDZ is initially saturated.

Note that the closed form expressions (3.6) and (3.7) give the variations of the stress state both on the Opalinus clay ($\Delta\sigma_r^{OPA}$, $\Delta\sigma_\theta^{OPA}$, $\Delta\sigma_z^{OPA}=0$ and $\Delta p^{*OPA}=0$) and on the EDZ ($\Delta\sigma_r^{EDZ}$, $\Delta\sigma_\theta^{EDZ}$ and $\Delta\sigma_z^{EDZ}$). Thus, in order to determine the stress state on the Opalinus clay or on the EDZ, we need to know the initial stress state both on the Opalinus clay

$((\sigma_r^{OPA})_0, (\sigma_\theta^{OPA})_0, (\sigma_z^{OPA})_0$ and $(p*^{OPA})_0)$ and on the EDZ $((\sigma_r^{EDZ})_0, (\sigma_\theta^{EDZ})_0$ and $(\sigma_z^{EDZ})_0)$ such that they satisfy the balance of momentum on both the Opalinus clay and on the EDZ and that $(p*^{OPA})_0$ only depends on z. However, we will leave the initial stress states undetermined and will only compute variations from these unknown stress states.

- **Boundary conditions at the test section end**. At $z=0$, water and/or air are injected into the test section at rates that are functions of time, according to the input protocol and are given as volume rates (volume of water or air per unit time). After transforming them into mass rates (mass of water or gas per unit time) $M_w(t)$ and $M_a(t)$, the boundary conditions at $z=0$ state the conservation of the mass of water and air that are injected.

$$
\begin{aligned}
M_w(t) = &-\pi(r_e^2 - r_i^2)\left\{\rho_l\left(\phi S_l D_l^w + d_l^l \frac{k_{rl}k}{\mu_l}\left|\frac{\partial p_l}{\partial z}\right|\right)\frac{\partial \omega_l^w}{\partial z} + \omega_l^w \rho_l \frac{k_{rl}k}{\mu_l}\frac{\partial p_l}{\partial z}\right\}\Bigg|_{z=0} \\
&-\pi(r_e^2 - r_i^2)\left\{\rho_g\left(\phi S_g D_g^w + d_g^l \frac{k_{rg}k}{\mu_g}\left|\frac{\partial p_g}{\partial z}\right|\right)\frac{\partial \omega_g^w}{\partial z} + \omega_g^w \rho_g \frac{k_{rg}k}{\mu_g}\frac{\partial p_g}{\partial z}\right\}\Bigg|_{z=0} \\
M_a(t) = &-\pi(r_e^2 - r_i^2)\left\{\rho_l\left(\phi S_l D_l^a + d_l^l \frac{k_{rl}k}{\mu_l}\left|\frac{\partial p_l}{\partial z}\right|\right)\frac{\partial \omega_l^a}{\partial z} + \omega_l^a \rho_l \frac{k_{rl}k}{\mu_l}\frac{\partial p_l}{\partial z}\right\}\Bigg|_{z=0} \\
&-\pi(r_e^2 - r_i^2)\left\{\rho_g\left(\phi S_g D_g^a + d_g^l \frac{k_{rg}k}{\mu_g}\left|\frac{\partial p_g}{\partial z}\right|\right)\frac{\partial \omega_g^a}{\partial z} + \omega_g^a \rho_g \frac{k_{rg}k}{\mu_g}\frac{\partial p_g}{\partial z}\right\}\Bigg|_{z=0}
\end{aligned}
\tag{3.22}
$$

- **Boundary conditions at the liner end**. They are based on discrete forms of Darcy's and Fick's laws, namely

$$
\begin{aligned}
&\left\{\rho_l\left(\phi S_l D_l^w + d_l^l \frac{k_{rl}k}{\mu_l}\left|\frac{\partial p_l}{\partial z}\right|\right)\frac{\partial \omega_l^w}{\partial z} + \omega_l^w \rho_l \frac{k_{rl}k}{\mu_l}\frac{\partial p_l}{\partial z}\right\}\Bigg|_{z=L} \\
&+\left\{\rho_g\left(\phi S_g D_g^w + d_g^l \frac{k_{rg}k}{\mu_g}\left|\frac{\partial p_g}{\partial z}\right|\right)\frac{\partial \omega_g^w}{\partial z} + \omega_g^w \rho_g \frac{k_{rg}k}{\mu_g}\frac{\partial p_g}{\partial z}\right\}\Bigg|_{z=L} \\
&\quad = \left\{\omega_l^w \rho_l \gamma_l \left(p_l^{ext} - p_l\right) + \beta_l \rho_l \left((\omega_l^w)^{ext} - \omega_l^w\right)\right\}\Big|_{z=L} \\
&\quad + \left\{\omega_g^w \rho_g \gamma_g \left(p_g^{ext} - p_g\right) + \beta_g \rho_g \left((\omega_g^w)^{ext} - \omega_g^w\right)\right\}\Big|_{z=L}
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
&\left\{\rho_l\left(\phi S_l D_l^w + d_l^l \frac{k_{rl}k}{\mu_l}\left|\frac{\partial p_l}{\partial z}\right|\right)\frac{\partial \omega_l^a}{\partial z} + \omega_l^a \rho_l \frac{k_{rl}k}{\mu_l}\frac{\partial p_l}{\partial z}\right\}\Bigg|_{z=L} \\
&+\left\{\rho_g\left(\phi S_g D_g^w + d_g^l \frac{k_{rg}k}{\mu_g}\left|\frac{\partial p_g}{\partial z}\right|\right)\frac{\partial \omega_g^a}{\partial z} + \omega_g^a \rho_g \frac{k_{rg}k}{\mu_g}\frac{\partial p_g}{\partial z}\right\}\Bigg|_{z=L} \\
&\quad = \left\{\omega_l^a \rho_l \gamma_l \left(p_l^{ext} - p_l\right) + \beta_l \rho_l \left((\omega_l^a)^{ext} - \omega_l^a\right)\right\}\Big|_{z=L} \\
&\quad + \left\{\omega_g^a \rho_g \gamma_g \left(p_g^{ext} - p_g\right) + \beta_g \rho_g \left((\omega_g^a)^{ext} - \omega_g^a\right)\right\}\Big|_{z=L}
\end{aligned}
\tag{3.24}
$$

where γ_l, γ_g, β_l and β_g are parameters, p_l^{ext} and p_g^{ext} are the prescribed values of pressures of the liquid and gas phases and $(\omega_l^w)^{ext}$, $(\omega_l^a)^{ext}$, $(\omega_g^w)^{ext}$ and $(\omega_g^a)^{ext}$ are the prescribed values of the mass fractions of water and dry air in the liquid and gas phases. Thus there are 10 parameters to be prescribed.

In the present case, the last 6 parameters may be written in terms of 3 parameters: the atmospheric pressure at the test site p^{atm}, the relative humidity of the gas phase at the test site h_r^{atm} and the mass concentration of dry air in the liquid phase at the test site $(\omega_l^a)^{ext}$.

$$p_g^{ext} = p^{atm} \tag{3.25}$$

$$(\omega_g^w \rho_g)^{ext} = \frac{M_w}{R(273.15+T)} p_{g0}^w(T) h_r^{ext} \tag{3.26}$$

$$(\omega_g^a \rho_g)^{ext} = \frac{M_a}{R(273.15+T)} \left(p_g^{ext} - p_{g0}^w(T) h_r^{ext} \right) \tag{3.27}$$

$$p_l^{ext} = p^{atm} \tag{3.28}$$

$$(\omega_l^w \rho_l)^{ext} = \rho_l [1 - (\omega_l^a)^{ext}] \tag{3.29}$$

$$(\omega_l^a \rho_l)^{ext} = \rho_l (\omega_l^a)^{ext} \tag{3.30}$$

Equations (3.25) and (3.28) state the assumption that the pressure of both the liquid and gas phases are the same as the atmospheric pressure at the test site. Equation (3.26) is the definition of relative humidity. Equation (3.27) follows from the expressions of p_g^w, ω_g^w and ρ_g from appendix A1 and the definition of relative humidity. Finally, equations (3.29) and (3.30) follow from the definition of $(\omega_l^a)^{ext}$, the constitutive law $\rho_l = const$ and the constraint $\omega_l^a + \omega_l^w = 1$. Consequently, the boundary conditions at $z = L$ depend (besides γ_l, γ_g, β_l and β_g) on the three parameters p^{atm}, h_r^{ext} and $(\omega_l^a)^{ext}$.

3.5 Test input protocol

The considered test input protocol consists of polygonal approximations to the records (measured by the appropriate test sensors) of (1) megapacker pressure (sensor M-PAMegapacker); (2) liquid phase volume injection rate (sensor M-Flow); and (3) gas phase volume injection rate (sensor M-GasFlow). These evolutions are shown in the Figure 3-2, with the polygonal approximations to them assumed in the modelling of the test.

The test input protocol applied to the test section end is stated as evolutions with time of the liquid phase volume injection rate $Q_l(t)$ (in ml/min) and gas phase volume injection rate $Q_g(t)$ (in mln/min). However, in the followed approach the boundary conditions considered at the test section end are stated as water mass injection rate $M_w(t)$ (in kg/s) and air mass injection rate $M_a(t)$ (in kg/s), since they are naturally associated to the balance equation of water mass and to the balance equation of air mass, respectively. In order to convert the input protocol to the considered boundary conditions, we proceed as follows

Since it has been assumed that the liquid density $\rho_l(T)$ does not depend neither on the liquid pressure nor on the amount of dissolved air, the water mass injection rate $M_w(t)$ (in kg/s) and air mass injection rate $M_a(t)$ (in kg/s) read

$$M_w(t)\frac{kg}{s} = Q_l(t)\frac{ml}{min}\times\frac{1min}{60s}\times\rho_l(T)\frac{kg}{m^3}\times\frac{1m^3}{1000l}\times\frac{1l}{1000ml}\times(1-\omega_{l\ ext}^{a})$$
$$+Q_g(t)\frac{mln}{min}\times\frac{1min}{60s}\times\frac{M_a\frac{kg}{mol}\times 0{,}1MPa}{R\frac{MPa\cdot m^3}{°K\cdot mol}\times(273.15+T)°K}\times\frac{1m^3n}{1000ln}\times\frac{1ln}{1000mln}\times\omega_{g\ ext}^{w} \tag{3.31}$$

$$M_a(t)\frac{kg}{s} = Q_l(t)\frac{ml}{min}\times\frac{1min}{60s}\times\rho_l(T)\frac{kg}{m^3}\times\frac{1m^3}{1000l}\times\frac{1l}{1000ml}\times\omega_{l\ ext}^{a}$$
$$+Q_g(t)\frac{mln}{min}\times\frac{1min}{60s}\times\frac{M_a\frac{kg}{mol}\times 0{,}1MPa}{R\frac{MPa\cdot m^3}{°K\cdot mol}\times(273.15+T)°K}\times\frac{1m^3n}{1000ln}\times\frac{1ln}{1000mln}\times(1-\omega_{g\ ext}^{w}) \tag{3.32}$$

where $Q_l(t)$ (in ml/min) is the volume injection rate, $Q_g(t)$ (in mln/min) the gas volume injection rate, $\omega_{l\ ext}^{a}$ is the air mass fraction in the liquid phase at the injection point and $\omega_{g\ ext}^{w}$ is the water mass fraction in the gas phase at the injection point, M_w and M_a are the molar mass of water and air and R is the constant of gases.

We compute $\omega_{l\ ext}^{a}$ using Henry's law at a gas pressure equal to the atmospheric pressure $p_{g\,ext} = p_{atm}$ (the atmospheric pressure at the experiment site)

$$\omega_{l\ ext}^{a} = \frac{M_a}{HM_w}p_{atm}, \tag{3.33}$$

where H is Henry's constant. We $\omega_{g\ ext}^{w}$ compute using the relation

$$\omega_{g\ ext}^{w} = \frac{M_w p_{g\ ext}^{w}}{M_w p_{g\ ext}^{w} + M_a p_{g\,ext}^{a}}, \tag{3.34}$$

where the vapour pressure $p_{g\ ext}^{w}$ is computed from the relative humidity $h_{r\,ext}$ using the psychrometric law and the partial pressure of dry air $p_{g\,ext}^{a}$ is computed from the gas pressure $p_{g\,ext} = p_{atm}$ (the atmospheric pressure at the experiment site) and $p_{g\ ext}^{w}$ using Dalton's law

$$p_{g\ ext}^{w} = h_{r\,ext}\,p_{g\,0}^{w}(T_{ext}) \quad \text{and} \quad p_{g\,ext}^{a} = p_{g\,ext} - p_{g\ ext}^{w}, \tag{3.35}$$

where $p_{g\,0}^{w}(T_{ext})$ is the water vapour pressure in equilibrium with a planar water surface at temperature T_{ext}.

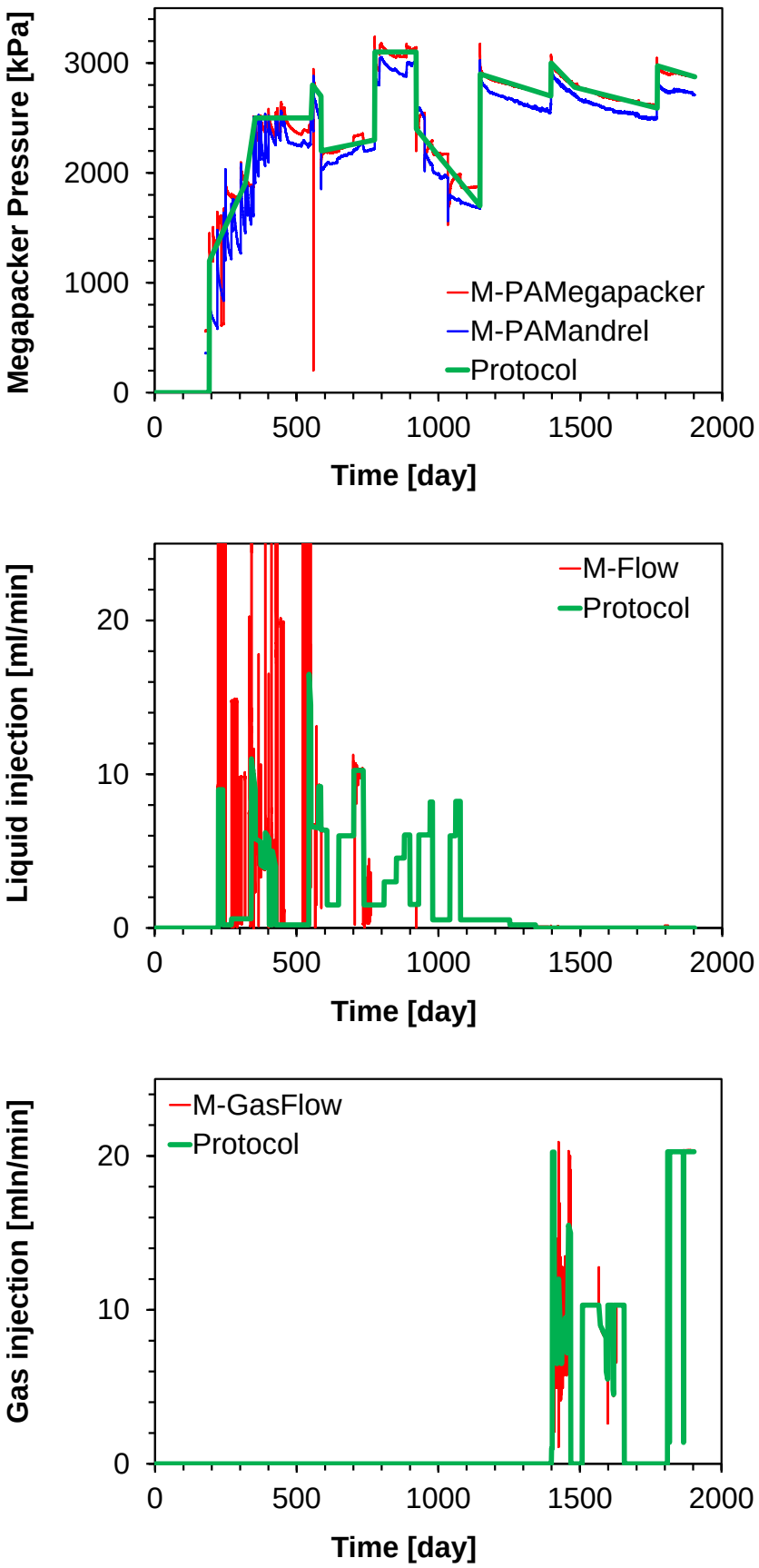

Figure 3-2. Test input protocol and polygonal approximations: evolution of (1) megapacker pressure; (2) liquid phase volume injection rate; and (3) gas phase volume injection rate.

4 Numerical solution using the FEM

In order to solve the coupled system of differential equations subject to the initial and boundary conditions presented in the previous paragraph, we will use the FEM with the Galerkin approach. In essence, this requires the following steps:

(1) determination of the weak form of the system of differential equations to solve;

(2) division of the spatial domain (spatial discretization) into a number of subdomains (finite elements), approximation of the unknown functions $p_l(z,t)$ and $p_g(z,t)$ using the values $p_l^I(t)$ and $p_g^I(t)$ of the unknown functions at a number of spatial points z^I (nodes) and an associated set of functions $w^I(z)$ (shape functions) and replacement of the system of differential equations by a system of ordinary differential equations (nodal equations), with the nodal values of the unknown functions $p_l^I(t)$ and $p_g^I(t)$ as unknowns and time t as the independent variable;

(3) integration of the system of ordinary differential equations using a time marching scheme that divides the time domain (time discretization) in time subdomains $[t^k, t^{k+1}]$ (time intervals) and in each of them approximates the unknown functions $p_l^I(t)$ and $p_g^I(t)$ using their values at the begin t^k and at the end t^{k+1} of the time interval and replaces the system of ordinary differential equations by a system of algebraic equations; and

(4) in each time interval $[t^k, t^{k+1}]$, calculation of the values $p_l^I(t^{k+1})$ and $p_g^I(t^{k+1})$ at the end of the time interval in terms of the values $p_l^I(t^k)$ and $p_g^I(t^k)$ at the beginning of the time interval by solving the associated system of algebraic equations.

4.1 Weak form of the problem

In order to numerically solve the coupled system of partial differential equations with the initial and boundary conditions, we first find its weak form.

4.1.1 Balance of water mass

We multiply that equation by an arbitrary test function $W_w(z)$ and we integrate it on the spatial domain $[0,L]$

$$\begin{aligned}
&\int_{[0,L]} W_w(z)\left\{\phi\frac{\partial}{\partial t}(\omega_l^w\rho_l S_l+\omega_g^w\rho_g S_g)+(\omega_l^w\rho_l S_l+\omega_g^w\rho_g S_g)\frac{\partial\varepsilon_{s,vol}}{\partial t}\right\}dz\\
&-\int_{[0,L]} W_w(z)\frac{\partial}{\partial z}\left\{\rho_l\left(\phi S_l D_l^w+d_l^l\frac{k_{rl}k}{\mu_l}\left|\frac{\partial p_l}{\partial z}\right|\right)\frac{\partial\omega_l^w}{\partial z}+\omega_l^w\rho_l\frac{k_{rl}k}{\mu_l}\frac{\partial p_l}{\partial z}\right\}dz\\
&-\int_{[0,L]} W_w(z)\frac{\partial}{\partial z}\left\{\rho_g\left(\phi S_g D_g^w+d_g^l\frac{k_{rg}k}{\mu_g}\left|\frac{\partial p_g}{\partial z}\right|\right)\frac{\partial\omega_g^w}{\partial z}+\omega_g^w\rho_g\frac{k_{rg}k}{\mu_g}\frac{\partial p_g}{\partial z}\right\}dz=0
\end{aligned}\tag{4.1}$$

Integrating by parts the second and third integrals, we get

$$
\begin{aligned}
&\int_{[0,L]} W_w(z)\left\{\phi\frac{\partial}{\partial t}(\omega_l^w\rho_l S_l+\omega_g^w\rho_g S_g)+(\omega_l^w\rho_l S_l+\omega_g^w\rho_g S_g)\frac{\partial\varepsilon_{s,vol}}{\partial t}\right\}dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_w(z)\right)\left(\rho_l\left(\phi S_l D_l^w+d_l^l\frac{k_{rl}k}{\mu_l}\left|\frac{\partial p_l}{\partial z}\right|\right)\frac{\partial\omega_l^w}{\partial z}+\omega_l^w\rho_l\frac{k_{rl}k}{\mu_l}\frac{\partial p_l}{\partial z}\right)dz\\
&-\left[W_w(z)\left(\rho_l\left(\phi S_l D_l^w+d_l^l\frac{k_{rl}k}{\mu_l}\left|\frac{\partial p_l}{\partial z}\right|\right)\frac{\partial\omega_l^w}{\partial z}+\omega_l^w\rho_l\frac{k_{rl}k}{\mu_l}\frac{\partial p_l}{\partial z}\right)\right]_{z=0}^{z=L}\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_w(z)\right)\left(\rho_g\left(\phi S_g D_g^w+d_g^l\frac{k_{rg}k}{\mu_g}\left|\frac{\partial p_g}{\partial z}\right|\right)\frac{\partial\omega_g^w}{\partial z}+\omega_g^w\rho_g\frac{k_{rg}k}{\mu_g}\frac{\partial p_g}{\partial z}\right)dz\\
&-\left[W_w(z)\left(\rho_g\left(\phi S_g D_g^w+d_g^l\frac{k_{rg}k}{\mu_g}\left|\frac{\partial p_g}{\partial z}\right|\right)\frac{\partial\omega_g^w}{\partial z}+\rho_g\omega_g^w\frac{k_{rg}k}{\mu_g}\frac{\partial p_g}{\partial z}\right)\right]_{z=0}^{z=L}=0
\end{aligned}
\tag{4.2}
$$

Using the boundary conditions, we get the weak form of the differential equation stating the balance of water mass.

$$
\begin{aligned}
&\int_{[0,L]} W_w(z)\left\{\phi\frac{\partial}{\partial t}(\omega_l^w\rho_l S_l+\omega_g^w\rho_g S_g)+(\omega_l^w\rho_l S_l+\omega_g^w\rho_g S_g)\frac{\partial\varepsilon_{s,vol}}{\partial t}\right\}dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_w(z)\right)\left(\rho_l\left(\phi S_l D_l^w+d_l^l\frac{k_{rl}k}{\mu_l}\left|\frac{\partial p_l}{\partial z}\right|\right)\frac{\partial\omega_l^w}{\partial z}+\omega_l^w\rho_l\frac{k_{rl}k}{\mu_l}\frac{\partial p_l}{\partial z}\right)dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_w(z)\right)\left(\rho_g\left(\phi S_g D_g^w+d_g^l\frac{k_{rg}k}{\mu_g}\left|\frac{\partial p_g}{\partial z}\right|\right)\frac{\partial\omega_g^w}{\partial z}+\omega_g^w\rho_g\frac{k_{rg}k}{\mu_g}\frac{\partial p_g}{\partial z}\right)dz\\
&-W_w(L)\left\{\omega_l^w\rho_l\gamma_l\left(p_l^{ext}-p_l\right)+\beta_l\rho_l\left((\omega_l^w)^{ext}-\omega_l^w\right)\right\}\Big|_{z=L}\\
&-W_w(L)\left\{\omega_g^w\rho_g\gamma_g\left(p_g^{ext}-p_g\right)+\beta_g\rho_g\left((\omega_g^w)^{ext}-\omega_g^w\right)\right\}\Big|_{z=L}\\
&-W_w(0)\frac{M_w(t)}{\pi(r_e^2-r_i^2)}=0
\end{aligned}
\tag{4.3}
$$

4.1.2 Balance of air mass

We multiply that equation by an arbitrary test function $W_a(z)$ and we integrate it on the spatial domain $[0,L]$

$$\begin{aligned}
&\int_{[0,L]} W_a(z)\left\{\phi\frac{\partial}{\partial t}(\omega_l^a\rho_l S_l+\omega_g^a\rho_g S_g)+(\omega_l^a\rho_l S_l+\omega_g^a\rho_g S_g)\frac{\partial\varepsilon_{s,vol}}{\partial t}\right\}dz\\
&-\int_{[0,L]} W_a(z)\frac{\partial}{\partial z}\left\{\rho_l\left(\phi S_l D_l^a+d_l^l\frac{k_{rl}k}{\mu_l}\left|\frac{\partial p_l}{\partial z}\right|\right)\frac{\partial\omega_l^a}{\partial z}+\omega_l^a\rho_l\frac{k_{rl}k}{\mu_l}\frac{\partial p_l}{\partial z}\right\}dz\\
&-\int_{[0,L]} W_a(z)\frac{\partial}{\partial z}\left\{\rho_g\left(\phi S_g D_g^a+d_g^l\frac{k_{rg}k}{\mu_g}\left|\frac{\partial p_g}{\partial z}\right|\right)\frac{\partial\omega_g^a}{\partial z}+\omega_g^w\rho_g\frac{k_{rg}k}{\mu_g}\frac{\partial p_g}{\partial z}\right\}dz=0
\end{aligned}\tag{4.4}$$

Integrating by parts the second and third integrals, we get

$$\begin{aligned}
&\int_{[0,L]} W_a(z)\left\{\phi\frac{\partial}{\partial t}(\omega_l^a\rho_l S_l+\omega_g^a\rho_g S_g)+(\omega_l^a\rho_l S_l+\omega_g^a\rho_g S_g)\frac{\partial\varepsilon_{s,vol}}{\partial t}\right\}dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_a(z)\right)\left(\rho_l\left(\phi S_l D_l^w+d_l^l\frac{k_{rl}k}{\mu_l}\left|\frac{\partial p_l}{\partial z}\right|\right)\frac{\partial\omega_l^a}{\partial z}+\omega_l^a\rho_l\frac{k_{rl}k}{\mu_l}\frac{\partial p_l}{\partial z}\right)dz\\
&-\left[W_a(z)\left(\rho_l\left(\phi S_l D_l^w+d_l^l\frac{k_{rl}k}{\mu_l}\left|\frac{\partial p_l}{\partial z}\right|\right)\frac{\partial\omega_l^a}{\partial z}+\omega_l^a\rho_l\frac{k_{rl}k}{\mu_l}\frac{\partial p_l}{\partial z}\right)\right]_{z=0}^{z=L}\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_a(z)\right)\left(\rho_g\left(\phi S_g D_g^w+d_g^l\frac{k_{rg}k}{\mu_g}\left|\frac{\partial p_g}{\partial z}\right|\right)\frac{\partial\omega_g^a}{\partial z}+\omega_g^a\rho_g\frac{k_{rg}k}{\mu_g}\frac{\partial p_g}{\partial z}\right)dz\\
&-\left[W_a(z)\left(\rho_g\left(\phi S_g D_g^w+d_g^l\frac{k_{rg}k}{\mu_g}\left|\frac{\partial p_g}{\partial z}\right|\right)\frac{\partial\omega_g^a}{\partial z}+\rho_g\omega_g^a\frac{k_{rg}k}{\mu_g}\frac{\partial p_g}{\partial z}\right)\right]_{z=0}^{z=L}=0
\end{aligned}\tag{4.5}$$

Using the boundary conditions, we get the weak form of the differential equation stating the balance of air mass.

$$\begin{aligned}
&\int_{[0,L]} W_a(z)\left\{\phi\frac{\partial}{\partial t}(\omega_l^a\rho_l S_l+\omega_g^a\rho_g S_g)+(\omega_l^a\rho_l S_l+\omega_g^a\rho_g S_g)\frac{\partial\varepsilon_{s,vol}}{\partial t}\right\}dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_a(z)\right)\left(\rho_l\left(\phi S_l D_l^w+d_l^l\frac{k_{rl}k}{\mu_l}\left|\frac{\partial p_l}{\partial z}\right|\right)\frac{\partial\omega_l^a}{\partial z}+\omega_l^a\rho_l\frac{k_{rl}k}{\mu_l}\frac{\partial p_l}{\partial z}\right)dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_a(z)\right)\left(\rho_g\left(\phi S_g D_g^w+d_g^l\frac{k_{rg}k}{\mu_g}\left|\frac{\partial p_g}{\partial z}\right|\right)\frac{\partial\omega_g^a}{\partial z}+\omega_g^a\rho_g\frac{k_{rg}k}{\mu_g}\frac{\partial p_g}{\partial z}\right)dz\\
&-W_a(L)\left\{\omega_l^a\rho_l\gamma_l\left(p_l^{ext}-p_l\right)+\beta_l\rho_l\left((\omega_l^a)^{ext}-\omega_l^a\right)\right\}\Big|_{z=L}\\
&-W_a(L)\left\{\omega_g^a\rho_g\gamma_g\left(p_g^{ext}-p_g\right)+\beta_g\rho_g\left((\omega_g^a)^{ext}-\omega_g^a\right)\right\}\Big|_{z=L}\\
&-W_a(0)\frac{M_a(t)}{\pi(r_e^2-r_i^2)}=0
\end{aligned}\tag{4.6}$$

A reformulation of these expressions that will be used in the subsequent numerical developments is shown in appendix A5.

4.2 Reduction to a system of ordinary differential equations

We divide the domain $I=[0,L]$ in adjacent finite elements I^e, such that $I=\bigcup_e I^e$ and we approximate the unknown functions $p_l(z,t)$ and $p_g(z,t)$ by using shape functions $w^I(z)$ ($I=1,\ldots,N$) such that $w^I(z^J)=\delta^{IJ}$ ($I,J=1,\ldots,N$), where z^I is the co-ordinate of node I, according to

$$p_l(z,t)=\sum_I w^I(z)p_l^I(t) \quad \text{and} \tag{4.7}$$

$$p_g(z,t)=\sum_I w^I(z)p_g^I(t)\,, \tag{4.8}$$

where $p_l^I(t)=p_l(z^I,t)$ and $p_g^I(t)=p_g(z^I,t)$.

In order to carry out this spatial approximation, we have used linear 1D finite elements with two nodes and one integration point. Each finite element is a segment with one node located at each end of the segment, the integration point is located at the center of the segment and the shape functions are linear. The linear shape functions, their derivatives and some related integrals may be found in the appendix A6

Following the Galerkin approach, we use each shape function $w^I(z)$ ($I=1,\ldots,N$) as the test functions $W_w(z)$ and $W_a(z)$. We get a system of $2N$ ordinary differential equations in the $2N$ unknowns $p_l^I(t)$ and $p_l^I(t)$ ($I=1,\ldots,N$)

$$\begin{aligned}
&\int_{[0,L]} w^I(z)\left\{\phi\frac{\partial}{\partial t}(\omega_l^w\rho_l S_l+\omega_g^w\rho_g S_g)+(\omega_l^w\rho_l S_l+\omega_g^w\rho_g S_g)\frac{\partial\varepsilon_{s,vol}}{\partial t}\right\}dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}w^I(z)\right)\left(\rho_l\left(\phi S_l D_l^w+d_l^l\frac{k_{rl}k}{\mu_l}\left|\frac{\partial p_l}{\partial z}\right|\right)\frac{\partial\omega_l^w}{\partial z}+\omega_l^w\rho_l\frac{k_{rl}k}{\mu_l}\frac{\partial p_l}{\partial z}\right)dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}w^I(z)\right)\left(\rho_g\left(\phi S_g D_g^w+d_g^l\frac{k_{rg}k}{\mu_g}\left|\frac{\partial p_g}{\partial z}\right|\right)\frac{\partial\omega_g^w}{\partial z}+\omega_g^w\rho_g\frac{k_{rg}k}{\mu_g}\frac{\partial p_g}{\partial z}\right)dz\\
&-w^I(L)\left\{\omega_l^w\rho_l\gamma_l\left(p_l^{ext}-p_l\right)+\beta_l\rho_l\left((\omega_l^w)^{ext}-\omega_l^w\right)\right\}\Big|_{z=L}\\
&-w^I(L)\left\{\omega_g^w\rho_g\gamma_g\left(p_g^{ext}-p_g\right)+\beta_g\rho_g\left((\omega_g^w)^{ext}-\omega_g^w\right)\right\}\Big|_{z=L}\\
&-w^I(0)\frac{M_w(t)}{\pi(r_e^2-r_i^2)}=0
\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
&\int_{[0,L]} w^I(z)\left\{\phi\frac{\partial}{\partial t}(\omega_l^a\rho_l S_l+\omega_g^a\rho_g S_g)+(\omega_l^a\rho_l S_l+\omega_g^a\rho_g S_g)\frac{\partial\varepsilon_{s,vol}}{\partial t}\right\}dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}w^I(z)\right)\left(\rho_l\left(\phi S_l D_l^w+d_l^l\frac{k_{rl}k}{\mu_l}\left|\frac{\partial p_l}{\partial z}\right|\right)\frac{\partial\omega_l^a}{\partial z}+\omega_l^a\rho_l\frac{k_{rl}k}{\mu_l}\frac{\partial p_l}{\partial z}\right)dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}w^I(z)\right)\left(\rho_g\left(\phi S_g D_g^w+d_g^l\frac{k_{rg}k}{\mu_g}\left|\frac{\partial p_g}{\partial z}\right|\right)\frac{\partial\omega_g^a}{\partial z}+\omega_g^a\rho_g\frac{k_{rg}k}{\mu_g}\frac{\partial p_g}{\partial z}\right)dz\\
&-w^I(L)\left\{\omega_l^a\rho_l\gamma_l\left(p_l^{ext}-p_l\right)+\beta_l\rho_l\left((\omega_l^a)^{ext}-\omega_l^a\right)\right\}\Big|_{z=L}\\
&-w^I(L)\left\{\omega_g^a\rho_g\gamma_g\left(p_g^{ext}-p_g\right)+\beta_g\rho_g\left((\omega_g^a)^{ext}-\omega_g^a\right)\right\}\Big|_{z=L}\\
&-w^I(0)\frac{M_a(t)}{\pi(r_e^2-r_i^2)}=0
\end{aligned}
\tag{4.10}
$$

The expression of this system of ordinary differential equations in terms of the auxiliary functions introduced when reformulating the equations defining the weak form of the problem is shown in the appendix A7. This system of ordinary differential equations may be written in the following compact form

$$
\boldsymbol{C}(\boldsymbol{X})\frac{d\boldsymbol{X}}{dt}+\boldsymbol{K}(\boldsymbol{X})\boldsymbol{X}=\boldsymbol{g}(\boldsymbol{X},t),
\tag{4.11}
$$

where $\boldsymbol{X}(t)$ is an unknown $2N$ vector function of t whose components are given by $X^{2I-1}(t)=p_l^I(t)$ and $X^{2I}(t)=p_g^I(t)$ ($I=1,\ldots,N$), $\boldsymbol{C}(\boldsymbol{X})$ and $\boldsymbol{K}(\boldsymbol{X})$ are $2N\times 2N$ matrices depending on $\boldsymbol{X}(t)$ and $\boldsymbol{g}(\boldsymbol{X},t)$ is a $2N$ vector depending on $\boldsymbol{X}(t)$ and t.

Taking into account the definition of the shape functions $w^I(z)$ ($I=1,\ldots,N$), element $[z^I,z^{I+1}]$ (between nodes I and $I+1$) contributes additively to the matrices $\boldsymbol{C}(\boldsymbol{X})$ and $\boldsymbol{K}(\boldsymbol{X})$ and to the vector $\boldsymbol{g}(\boldsymbol{X},t)$. More precisely, if we denote the contributions from element $[z^I,z^{I+1}]$ to these matrices by

$$
\boldsymbol{C}_{[z^I,z^{I+1}]}=\begin{vmatrix}
C_{w,I}^{I,l}(p_l^K,p_g^L) & C_{w,I}^{I,g}(p_l^K,p_g^L) & C_{w,I+1}^{I,l}(p_l^K,p_g^L) & C_{w,I+1}^{I,g}(p_l^K,p_g^L)\\
C_{a,I}^{I,l}(p_l^K,p_g^L) & C_{a,I}^{I,g}(p_l^K,p_g^L) & C_{a,I+1}^{I,l}(p_l^K,p_g^L) & C_{a,I+1}^{I,g}(p_l^K,p_g^L)\\
C_{w,I}^{I+1,l}(p_l^K,p_g^L) & C_{w,I}^{I+1,g}(p_l^K,p_g^L) & C_{w,I+1}^{I+1,l}(p_l^K,p_g^L) & C_{w,I+1}^{I+1,g}(p_l^K,p_g^L)\\
C_{a,I}^{I+1,l}(p_l^K,p_g^L) & C_{a,I}^{I+1,g}(p_l^K,p_g^L) & C_{a,I+1}^{I+1,l}(p_l^K,p_g^L) & C_{a,I+1}^{I+1,g}(p_l^K,p_g^L)
\end{vmatrix},
\tag{4.12}
$$

$$
\boldsymbol{K}_{[z^I,z^{I+1}]}=\begin{vmatrix}
K_{w,I}^{I,l}(p_l^K,p_g^L) & K_{w,I}^{I,g}(p_l^K,p_g^L) & K_{w,I+1}^{I,l}(p_l^K,p_g^L) & K_{w,I+1}^{I,g}(p_l^K,p_g^L)\\
K_{a,I}^{I,l}(p_l^K,p_g^L) & K_{a,I}^{I,g}(p_l^K,p_g^L) & K_{a,I+1}^{I,l}(p_l^K,p_g^L) & K_{a,I+1}^{I,g}(p_l^K,p_g^L)\\
K_{w,I}^{I+1,l}(p_l^K,p_g^L) & K_{w,I}^{I+1,g}(p_l^K,p_g^L) & K_{w,I+1}^{I+1,l}(p_l^K,p_g^L) & K_{w,I+1}^{I+1,g}(p_l^K,p_g^L)\\
K_{a,I}^{I+1,l}(p_l^K,p_g^L) & K_{a,I}^{I+1,g}(p_l^K,p_g^L) & K_{a,I+1}^{I+1,l}(p_l^K,p_g^L) & K_{a,I+1}^{I+1,g}(p_l^K,p_g^L)
\end{vmatrix}
\quad\text{and}
\tag{4.13}
$$

$$\boldsymbol{g}_{[z^I,z^{I+1}]} = \begin{vmatrix} g_w^I(p_l^K, p_g^L, t) \\ g_a^I(p_l^K, p_g^L, t) \\ g_w^{I+1}(p_l^K, p_g^L, t) \\ g_a^{I+1}(p_l^K, p_g^L, t) \end{vmatrix}, \tag{4.14}$$

where the expression of their elements in terms of the auxiliary functions introduced when reformulating the equations defining the weak form of the problem are shown in the appendix A8 and, due to the shape functions used, they have the following symmetries

$$\begin{aligned}
&C_{w,I}^{I,l}(p_l^K, p_g^L) = C_{w,I+1}^{I,l}(p_l^K, p_g^L) = C_{w,I}^{I+1,l}(p_l^K, p_g^L) = C_{w,I+1}^{I+1,l}(p_l^K, p_g^L) \\
&C_{w,I}^{I,g}(p_l^K, p_g^L) = C_{w,I+1}^{I,g}(p_l^K, p_g^L) = C_{w,I}^{I+1,g}(p_l^K, p_g^L) = C_{w,I+1}^{I+1,g}(p_l^K, p_g^L) \\
&C_{a,I}^{I,l}(p_l^K, p_g^L) = C_{a,I+1}^{I,l}(p_l^K, p_g^L) = C_{a,I}^{I+1,l}(p_l^K, p_g^L) = C_{a,I+1}^{I+1,l}(p_l^K, p_g^L) \\
&C_{a,I}^{I,g}(p_l^K, p_g^L) = C_{a,I+1}^{I,g}(p_l^K, p_g^L) = C_{a,I}^{I+1,g}(p_l^K, p_g^L) = C_{a,I+1}^{I+1,g}(p_l^K, p_g^L)
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
&K_{w,I}^{I,l}(p_l^K, p_g^L) = -K_{w,I+1}^{I,l}(p_l^K, p_g^L) = -K_{w,I}^{I+1,l}(p_l^K, p_g^L) = K_{w,I+1}^{I+1,l}(p_l^K, p_g^L) \\
&K_{w,I}^{I,g}(p_l^K, p_g^L) = -K_{w,I+1}^{I,g}(p_l^K, p_g^L) = -K_{w,I}^{I+1,g}(p_l^K, p_g^L) = K_{w,I+1}^{I+1,g}(p_l^K, p_g^L) \\
&K_{a,I}^{I,l}(p_l^K, p_g^L) = -K_{a,I+1}^{I,l}(p_l^K, p_g^L) = -K_{a,I}^{I+1,l}(p_l^K, p_g^L) = K_{a,I+1}^{I+1,l}(p_l^K, p_g^L) \\
&K_{a,I}^{I,g}(p_l^K, p_g^L) = -K_{a,I+1}^{I,g}(p_l^K, p_g^L) = -K_{a,I}^{I+1,g}(p_l^K, p_g^L) = K_{a,I+1}^{I+1,g}(p_l^K, p_g^L)
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
&g_w^I(p_l^K, p_g^L, t) = g_w^{I+1}(p_l^K, p_g^L, t) \\
&g_a^I(p_l^K, p_g^L, t) = g_a^{I+1}(p_l^K, p_g^L, t)
\end{aligned} \tag{4.17}$$

then the elements (L,M) $(L,M=1,2,3,4)$ of the 4×4 matrices $\boldsymbol{C}_{[z^I,z^{I+1}]}$ and $\boldsymbol{K}_{[z^I,z^{I+1}]}$ are added to the elements $(2(I-1)+L, 2(I-1)+M)$ of the $2N \times 2N$ matrices $\boldsymbol{C}(\boldsymbol{X})$ and $\boldsymbol{K}(\boldsymbol{X})$ and the elements (L) $(L=1,2,3,4)$ of the 4 vector $\boldsymbol{g}_{[z^I,z^{I+1}]}$ are added to the elements $(2(I-1)+L)$ of the $2N$ vector $\boldsymbol{g}(\boldsymbol{X},t)$. Moreover, due to the boundary conditions, node 1 (corresponding to the test section end) contributes additively to the vector $\boldsymbol{g}(\boldsymbol{X},t)$ and node N (corresponding to the liner end) contributes additively to the vector $\boldsymbol{g}(\boldsymbol{X},t)$ and to the matrix $\boldsymbol{K}(\boldsymbol{X})$. More precisely, if we denote the contributions from nodes 1 and N to these matrices by

$$\boldsymbol{g}_{z^1} = \begin{vmatrix} g_w^1(p_l^K, p_g^L, t) \\ g_a^1(p_l^K, p_g^L, t) \end{vmatrix}, \tag{4.18}$$

$$\boldsymbol{g}_{z^N} = \begin{vmatrix} g_w^N(p_l^K, p_g^L, t) \\ g_a^N(p_l^K, p_g^L, t) \end{vmatrix} \quad \text{and} \tag{4.19}$$

$$\boldsymbol{K}_{z^N} = \begin{vmatrix} K_{w,N}^{N,l}(p_l^K, p_g^L) & K_{w,N}^{N,g}(p_l^K, p_g^L) \\ K_{a,N}^{N,l}(p_l^K, p_g^L) & K_{a,N}^{N,g}(p_l^K, p_g^L) \end{vmatrix}, \tag{4.20}$$

where the expression of their elements in terms of the auxiliary functions introduced when reformulating the equations defining the weak form of the problem are shown in the appendix

A8, then the elements (L,M) $(L,M=1,2)$ of the 2×2 matrix $\boldsymbol{K}_{z^N}$ are added to the elements $(2(N-1)+L, 2(N-1)+M)$ of the $2N\times 2N$ matrix $\boldsymbol{K}(\boldsymbol{X})$, the elements (L) $(L=1,2)$ of the 2 vector of the vector $\boldsymbol{g}_{z^1}$ are added to the elements (L) of the $2N$ vector $\boldsymbol{g}(\boldsymbol{X},t)$ and the elements (L) $(L=1,2)$ of the 2 vector $\boldsymbol{g}_{z^N}$ are added to the elements (L) $(2(N-1)+L)$ of the $2N$ vector $\boldsymbol{g}(\boldsymbol{X},t)$. As a result, the $2N\times 2N$ matrices $\boldsymbol{C}(\boldsymbol{X})$ and $\boldsymbol{K}(\boldsymbol{X})$ are 4×4-block 3-diagonal and, consequently, 7-diagonal. The 4×4 matrices $\boldsymbol{C}_{[z^I,z^{I+1}]}$ and $\boldsymbol{K}_{[z^I,z^{I+1}]}$ ($I=1,\ldots,N$), the 2 vectors $\boldsymbol{g}_{[z^I,z^{I+1}]}$ ($I=1,\ldots,N$), the 2×2 matrix $\boldsymbol{K}_{z^N}$ and the 2 vectors $\boldsymbol{g}_{z^1}$ and $\boldsymbol{g}_{z^N}$ are detailed in the appendix A8.

The elements of the element matrices $\boldsymbol{C}_{[z^I,z^{I+1}]}$ and $\boldsymbol{K}_{[z^I,z^{I+1}]}$ and of the element vector $\boldsymbol{g}_{[z^I,z^{I+1}]}$, corresponding to element $[z^I,z^{I+1}]$ are integrals over the $[z^I,z^{I+1}]$. These integrals have been computed numerically, using the following approximation

$$\begin{aligned}&\int_{[z^I,z^{I+1}]} f\left[p_l(z,t), p_g(z,t), \frac{\partial p_l}{\partial z}(z,t), \frac{\partial p_g}{\partial z}(z,t), t\right]dz\\ &\simeq (z^{I+1}-z^I)\, f\left[p_l\left(\frac{z^{I+1}+z^I}{2}\right), p_g\left(\frac{z^{I+1}+z^I}{2}\right), \frac{\partial p_l}{\partial z}\left(\frac{z^{I+1}+z^I}{2}\right), \frac{\partial p_g}{\partial z}\left(\frac{z^{I+1}+z^I}{2}\right), t\right]\end{aligned} \tag{4.21}$$

If we use the approximations $p_l(z,t)=\sum_I w^I(z)p_l^I(t)$ and $p_g(z,t)=\sum_I w^I(z)p_g^I(t)$, where $w^I(z)$ $(1<I<N)$ is defined in appendix A6 , we get

$$\begin{aligned}&\int_{[z^I,z^{I+1}]} f\left[p_l(z,t), p_g(z,t), \frac{\partial p_l}{\partial z}(z,t), \frac{\partial p_g}{\partial z}(z,t), t\right]dz\\ &\simeq (z^{I+1}-z^I)\, f\left[\frac{p_l^{I+1}(t)+p_l^I(t)}{2}, \frac{p_g^{I+1}(t)+p_g^I(t)}{2}, \frac{p_l^{I+1}(t)-p_l^I(t)}{z^{I+1}-z^I}, \frac{p_g^{I+1}(t)-p_g^I(t)}{z^{I+1}-z^I}, t\right]\end{aligned} \tag{4.22}$$

Therefore, the integration uses an integration point located at $(z^{I+1}+z^I)/2$, that is, at the middle of the element $[z^I,z^{I+1}]$,.

4.3 Integration of the system of ordinary differential equations

We integrate this system of ordinary differential equations using a time marching scheme. We will consider a series of time instants t^k $(k=0,1,2,\ldots)$ such that $t^{k+1}>t^k$ $(k=0,1,2,\ldots)$ and t^0 is the initial time (the procedure employed to determine these time instants will be explained later). Furthermore, on each time interval $t^k\le t\le t^{k+1}$ $(k=0,1,2,\ldots)$, we will approximate the vector function $\boldsymbol{X}(t)$ linearly with respect to time using $\boldsymbol{X}(t^k)$ and $\boldsymbol{X}(t^{k+1})$.

$$\boldsymbol{X}(t)=\frac{t^{k+1}-t}{t^{k+1}-t^k}\boldsymbol{X}(t^k)+\frac{t-t^k}{t^{k+1}-t^k}\boldsymbol{X}(t^{k+1}) \qquad (t^k\le t\le t^{k+1},\ k=0,1,2,\ldots). \tag{4.23}$$

Thus, on each time interval $t^k \le t \le t^{k+1}$ ($k = 0,1,2,\dots$), the time derivative of $\boldsymbol{X}(t)$ reads

$$\frac{d\boldsymbol{X}}{dt}(t) = \frac{\boldsymbol{X}(t^{k+1}) - \boldsymbol{X}(t^k)}{t^{k+1} - t^k} \qquad (t^k \le t \le t^{k+1},\ k = 0,1,2,\dots). \tag{4.24}$$

With these assumptions, at each time t ($t^k \le t \le t^{k+1}$, $k = 0,1,2,\dots$), the system of ordinary differential equations transforms into a system of algebraic equations in the unknowns $\boldsymbol{X}(t^k)$ ($k = 1,2,\dots$), since $\boldsymbol{X}(t^0)$ is known.

$$\begin{aligned}
&\boldsymbol{C}\left(\frac{t^{k+1}-t}{t^{k+1}-t^k}\boldsymbol{X}(t^k) + \frac{t-t^k}{t^{k+1}-t^k}\boldsymbol{X}(t^{k+1})\right)\left(\frac{\boldsymbol{X}(t^{k+1})-\boldsymbol{X}(t^k)}{t^{k+1}-t^k}\right)\\
&+\boldsymbol{K}\left(\frac{t^{k+1}-t}{t^{k+1}-t^k}\boldsymbol{X}(t^k) + \frac{t-t^k}{t^{k+1}-t^k}\boldsymbol{X}(t^{k+1})\right)\left(\frac{t^{k+1}-t}{t^{k+1}-t^k}\boldsymbol{X}(t^k) + \frac{t-t^k}{t^{k+1}-t^k}\boldsymbol{X}(t^{k+1})\right) \qquad .\\
&= \boldsymbol{g}\left(\left(\frac{t^{k+1}-t}{t^{k+1}-t^k}\boldsymbol{X}(t^k) + \frac{t-t^k}{t^{k+1}-t^k}\boldsymbol{X}(t^{k+1})\right), t\right) \qquad (t^k \le t \le t^{k+1},\quad k = 0,1,2,\dots)
\end{aligned} \tag{4.25}$$

On each time interval $t^k \le t \le t^{k+1}$ ($k = 0,1,2,\dots$), we consider the change of independent variable $\xi^k(t) = (t - t^k)/(t^{k+1} - t^k)$ ($t^k \le t \le t^{k+1}$) whose inverse is $t(\xi^k) = (1-\xi^k)t^k + \xi^k t^{k+1}$ ($0 \le \xi^k \le 1$). Using the notation $t^{k+\xi} \equiv t(\xi^k)$ and $\boldsymbol{X}^{k+\xi} \equiv \boldsymbol{X}(t(\xi^k))$ ($k = 0,1,2,\dots$, $0 \le \xi^k \le 1$), we have

$$\boldsymbol{X}^{k+\xi} = (1-\xi^k)\boldsymbol{X}^k + \xi^k \boldsymbol{X}^{k+1} \quad (0 \le \xi^k \le 1,\ k = 0,1,2,\dots), \tag{4.26}$$

$$\frac{d\boldsymbol{X}^{k+\xi}}{dt} = \frac{\boldsymbol{X}^{k+1} - \boldsymbol{X}^k}{t^{k+1} - t^k} \qquad (0 \le \xi^k \le 1,\ k = 0,1,2,\dots). \tag{4.27}$$

where $d\boldsymbol{X}^{k+\xi}/dt \equiv (d\boldsymbol{X}/dt)(t(\xi^k))$ denotes the time derivative of $\boldsymbol{X}(t)$ evaluated at $t(\xi^k)$. With this notation, at time ξ^k ($0 \le \xi^k \le 1$), the system of algebraic equations reads

$$\boldsymbol{C}(\boldsymbol{X}^{k+\xi})\frac{\boldsymbol{X}^{k+1} - \boldsymbol{X}^k}{t^{k+1} - t^k} + \boldsymbol{K}(\boldsymbol{X}^{k+\xi})\boldsymbol{X}^{k+\xi} - \boldsymbol{g}\left(\boldsymbol{X}^{k+\xi}, t^{k+\xi}\right) = 0 \qquad (0 \le \xi^k \le 1,\ k = 0,1,2,\dots). \tag{4.28}$$

In general, it is not possible to find $\boldsymbol{X}(t^k)$ ($k = 1,2,\dots$) such that the functions (4.26) satisfy the system of algebraic equations (4.28) associated to time ξ^k in the time interval $t^k \le t \le t^{k+1}$ ($k = 0,1,2,\dots$) for all values of ξ^k such that $0 \le \xi^k \le 1$, because there are infinitely many such systems.

A possibility that is often used is to require that the equality holds for a certain value $0 \le \theta \le 1$ of ξ^k denoting a time in each time interval $t^k \le t \le t^{k+1}$ ($k = 0,1,2,\dots$), that is, we consider the following series of systems of algebraic equations

$$\boldsymbol{C}(\boldsymbol{X}^{k+\theta})\frac{\boldsymbol{X}^{k+1} - \boldsymbol{X}^k}{t^{k+1} - t^k} + \boldsymbol{K}(\boldsymbol{X}^{k+\theta})\boldsymbol{X}^{k+\theta} - \boldsymbol{g}\left(\boldsymbol{X}^{k+\theta}, t^{k+\theta}\right) = 0 \qquad (k = 0,1,2,\dots). \tag{4.29}$$

Note that the extreme values $\theta = 0$ and $\theta = 1$ correspond, respectively, to the explicit and implicit approaches. If the vector $\boldsymbol{X}^k$ is known, we compute the vector $\boldsymbol{X}^{k+1}$ by solving the k-th system of algebraic equations. Therefore, since the vector $\boldsymbol{X}^0$ is known from the initial conditions, we have to successively solve the above series of systems of algebraic equations.

A somewhat more general approach is to consider the satisfaction of the series of systems of algebraic equations

$$\boldsymbol{C}(\boldsymbol{X}^{k+\varepsilon})\frac{\boldsymbol{X}^{k+1}-\boldsymbol{X}^k}{t^{k+1}-t^k}+\boldsymbol{K}(\boldsymbol{X}^{k+\varepsilon})\boldsymbol{X}^{k+\theta}-\boldsymbol{g}(\boldsymbol{X}^{k+\varepsilon},t^{k+\theta})=0 \qquad (k=0,1,2,\ldots), \tag{4.30}$$

where $0 \le \varepsilon \le \theta \le 1$. Note that if $\varepsilon = \theta$ we recover the previous approach and if $\varepsilon = 0$ the system of algebraic equations is linear in the unknowns $\boldsymbol{X}^{k+1}$

$$\boldsymbol{C}(\boldsymbol{X}^{k})\frac{\boldsymbol{X}^{k+1}-\boldsymbol{X}^k}{t^{k+1}-t^k}+\boldsymbol{K}(\boldsymbol{X}^{k})\boldsymbol{X}^{k+\theta}-\boldsymbol{g}(\boldsymbol{X}^{k},t^{k+\theta})=0 \qquad (k=0,1,2,\ldots). \tag{4.31}$$

Therefore, the value ε may be viewed as a measure of the non-linearity of the system of algebraic equations. If we introduce the definition

$$\boldsymbol{F}(\boldsymbol{X}^{k+1}) \equiv \boldsymbol{C}(\boldsymbol{X}^{k+\varepsilon})\frac{\boldsymbol{X}^{k+1}-\boldsymbol{X}^k}{t^{k+1}-t^k}+\boldsymbol{K}(\boldsymbol{X}^{k+\varepsilon})\boldsymbol{X}^{k+\theta}-\boldsymbol{g}(\boldsymbol{X}^{k+\varepsilon},t^{k+\theta}) \qquad (k=0,1,2,\ldots), \tag{4.32}$$

then the series of systems of algebraic equations (4.30) reads

$$\boldsymbol{F}(\boldsymbol{X}^{k+1})=0 \qquad (k=0,1,2,\ldots). \tag{4.33}$$

As before, if the vector $\boldsymbol{X}^k$ is known, we compute the vector $\boldsymbol{X}^{k+1}$ by solving the k-th system of algebraic equations. Therefore, since the vector $\boldsymbol{X}^0$ is known from the initial conditions, we have to successively solve the above series of systems of algebraic equations.

4.4 Solution of the systems of algebraic equations

In order to solve the system of algebraic equations

$$\boldsymbol{F}(\boldsymbol{X}^{k+1})=0 \tag{4.34}$$

for a certain k ($k=0,1,2,\ldots$), one possibility is to use the Newton-Raphson method, which is an iterative procedure, whereby an initial estimation of the solution is successively updated until a certain criterion of convergence is satisfied. However, we have used a modification of this method. We first describe the Newton-Raphson method and afterwards we describe the modified version that has been used.

4.4.1 Newton-Raphson method

It is defined by the following algorithm

(1) **Initial estimation of the solution ($\boldsymbol{X}^{k+1,0}$).** Define an initial estimation of the solution.

(2) **New estimation of the solution ($\boldsymbol{X}^{k+1,i} \rightarrow \boldsymbol{X}^{k+1,i+1}$).** Knowing an estimation of the solution $\boldsymbol{X}^{k+1,i}$ (either from the initial value if $i=0$ or from the previous estimation if $i>0$), a new estimation of the solution $\boldsymbol{X}^{k+1,i+1}$ is found by solving the following system of linear algebraic equations

$$\frac{\partial \boldsymbol{F}(\boldsymbol{X}^{k+1,i})}{\partial \boldsymbol{X}^{k+1}}(\boldsymbol{X}^{k+1,i+1}-\boldsymbol{X}^{k+1,i}) = -\boldsymbol{F}(\boldsymbol{X}^{k+1,i}), \tag{4.35}$$

where $\partial \boldsymbol{F}(\boldsymbol{X}^{k+1,i})/\partial \boldsymbol{X}^{k+1}$ denotes the Jacobian matrix of the derivatives of the vector function $\boldsymbol{F}(\boldsymbol{X}^{k+1})$ with respect to the vector variable $\boldsymbol{X}^{k+1}$ evaluated at $\boldsymbol{X}^{k+1}=\boldsymbol{X}^{k+1,i}$.

(3) C**heck of the current estimation of the solution.** If the current estimation of the solution satisfies a certain convergence criterion, it is taken to be the solution of the system of algebraic equations and we are done, otherwise we go to step (2) to find a new estimation of the solution.

4.4.2 Implemented solution algorithm

The solution algorithm used is based on the Newton-Raphson method, whereby its three steps have been defined as follows.

Step (1). In this step, an initial estimation of the solution must be made. We compute it as follows

$$\boldsymbol{X}^{k+1,0} = \boldsymbol{X}^{k} \qquad (k=0) \quad \text{or} \tag{4.36}$$

$$\boldsymbol{X}^{k+1,0} = \boldsymbol{X}^{k} + \frac{t^{k+1}-t^{k}}{t^{k}-t^{k-1}}(\boldsymbol{X}^{k}-\boldsymbol{X}^{k-1}) \quad (k>0), \tag{4.37}$$

where $\boldsymbol{X}^{k}$ and $\boldsymbol{X}^{k-1}$ are the values computed previously when solving the systems of algebraic equations $\boldsymbol{F}(\boldsymbol{X}^{k})=0$ and $\boldsymbol{F}(\boldsymbol{X}^{k-1})=0$, respectively.

Step (2). In this step, the Jacobian matrix $\partial \boldsymbol{F}(\boldsymbol{X}^{k+1,i})/\partial \boldsymbol{X}^{k+1}$ must be computed, which in our case reads

$$\begin{aligned}\frac{\partial \boldsymbol{F}(\boldsymbol{X}^{k+1,i})}{\partial \boldsymbol{X}^{k+1}} = \; & \mathbf{C}\left(\mathbf{X}^{k+\varepsilon,i}\right)\frac{1}{t^{k+1}-t^{k}} + \mathbf{K}\left(\mathbf{X}^{k+\varepsilon,i}\right)\theta \\ & + \frac{\partial \mathbf{C}\left(\mathbf{X}^{k+\varepsilon,i}\right)}{\partial \mathbf{X}}\varepsilon\frac{\mathbf{X}^{k+1,i}-\mathbf{X}^{k}}{t^{k+1}-t^{k}} + \frac{\partial \mathbf{K}\left(\mathbf{X}^{k+\varepsilon,i}\right)}{\partial \mathbf{X}}\varepsilon\mathbf{X}^{k+\theta,i} - \frac{\partial \mathbf{g}\left(\mathbf{X}^{k+\varepsilon,i},t^{k+\theta}\right)}{\partial \mathbf{X}}\varepsilon\end{aligned}, \tag{4.38}$$

where $\mathbf{X}^{k+\varepsilon,i} = (1-\varepsilon)\mathbf{X}^k + \varepsilon\mathbf{X}^{k+1,i}$, $\partial \boldsymbol{C}(\boldsymbol{X}^{k+\varepsilon,i})/\partial \boldsymbol{X}$ and $\partial \boldsymbol{K}(\boldsymbol{X}^{k+\varepsilon,i})/\partial \boldsymbol{X}$ denote the partial derivatives of the matrix functions $\boldsymbol{C}(\boldsymbol{X})$ and $\boldsymbol{K}(\boldsymbol{X})$ with respect to the vector variable $\boldsymbol{X}$ evaluated at $\boldsymbol{X} = \boldsymbol{X}^{k+\varepsilon,i}$ and $\partial \boldsymbol{g}(\boldsymbol{X}^{k+\varepsilon,i}, t^{k+\theta})/\partial \boldsymbol{X}$ denotes the partial derivatives of the vector function $\boldsymbol{g}(\boldsymbol{X}, t)$ with respect to the vector variable $\boldsymbol{X}$ evaluated at $\boldsymbol{X} = \boldsymbol{X}^{k+\varepsilon,i}$ and $t = t^{k+\theta}$. Note that if $\varepsilon = 0$, this reduces to

$$\frac{\partial \boldsymbol{F}(\boldsymbol{X}^{k+1,i})}{\partial \boldsymbol{X}^{k+1}} = \mathbf{C}\left(\mathbf{X}^{k,i}\right)\frac{1}{t^{k+1}-t^k} + \mathbf{K}\left(\mathbf{X}^{k,i}\right)\theta, \tag{4.39}$$

which is independent of $\boldsymbol{X}^{k+1,i}$.

This matrix is quite involved and its computation is deemed to be quite time demanding. In order to overcome these problems, we approximate the Jacobian matrix by neglecting the derivatives of the nonlinear terms (we do not derive a term with respect to $\boldsymbol{X}^{k+1}$ if it is multiplied by ε), so that the Jacobian matrix $\partial \boldsymbol{F}(\boldsymbol{X}^{k+1})/\partial \boldsymbol{X}^{k+1}$ that has been used reads

$$\frac{\partial \boldsymbol{F}(\boldsymbol{X}^{k+1,i})}{\partial \boldsymbol{X}^{k+1}} = \mathbf{C}\left(\mathbf{X}^{k+\varepsilon,i}\right)\frac{1}{t^{k+1}-t^k} + \mathbf{K}\left(\mathbf{X}^{k+\varepsilon,i}\right)\theta, \tag{4.40}$$

which depends on $\boldsymbol{X}^{k+1,i}$, because $\mathbf{X}^{k+\varepsilon,i} = (1-\varepsilon)\mathbf{X}^k + \varepsilon\mathbf{X}^{k+1,i}$. This matrix is 7-diagonal and, if $\varepsilon \neq 0$, it is not symmetric. Therefore, the system of linear algebraic equations in the unknowns $\boldsymbol{X}^{k+1,i+1}$ to solve reads

$$\begin{aligned}\left[\mathbf{C}\left(\mathbf{X}^{k+\varepsilon,i}\right)\frac{1}{t^{k+1}-t^k} + \mathbf{K}\left(\mathbf{X}^{k+\varepsilon,i}\right)\theta\right](\boldsymbol{X}^{k+1,i+1} - \boldsymbol{X}^{k+1,i}) \\ = -\boldsymbol{C}(\boldsymbol{X}^{k+\varepsilon,i})\frac{\boldsymbol{X}^{k+1,i} - \boldsymbol{X}^k}{t^{k+1}-t^k} - \boldsymbol{K}(\boldsymbol{X}^{k+\varepsilon,i})\boldsymbol{X}^{k+\theta,i} + \boldsymbol{g}(\boldsymbol{X}^{k+\varepsilon,i}, t^{k+\theta})\end{aligned}, \tag{4.41}$$

where $\mathbf{X}^{k+\theta,i} = (1-\theta)\mathbf{X}^k + \theta\mathbf{X}^{k+1,i}$. This system is solved by means of an algorithm for inversion of nonsymmetrical band matrices, based on triangularisation and backsubstitution.

Step (3). In this step, the current estimation $\boldsymbol{X}^{k+1,i}$ is accepted as a solution if all the following conditions are satisfied

$$\max_{I=1,\ldots,N}\left|(X^{2I-1})^{k+1,i} - (X^{2I-1})^{k+1,i-1}\right| < tol_p_l_SOLV \quad \text{and} \tag{4.42}$$

$$\max_{I=1,\ldots,N}\left|(X^{2I})^{k+1,i} - (X^{2I})^{k+1,i-1}\right| < tol_p_g_SOLV, \tag{4.43}$$

where, for $I = 1,\ldots,N$, $(X^{2I-1})^{k+1,i}$ and $(X^{2I})^{k+1,i}$ are the components $2I-1$ and $2I$ of the vector $\boldsymbol{X}^{k+1,i}$ (the current estimations of $p_l^I(t^{k+1}) = p_l(z^I, t^{k+1})$ and $p_g^I(t^{k+1}) = p_g(z^I, t^{k+1})$), $(X^{2I-1})^{k+1,i-1}$ and $(X^{2I})^{k+1,i-1}$ are the corresponding values of the preceding estimation, $tol_p_l_SOLV$ is the tolerance for corrections of the liquid phase pressure p_l at the nodes and $tol_p_g_SOLV$ is the tolerance for corrections of the gas phase pressure p_g at the nodes. Furthermore, in order to prevent too many iterations or too large deviations from the

tolerances set for the liquid phase pressure p_l and for the gas phase pressure p_g, in step (3) the process is aborted if at least one of the following conditions is satisfied

$$i > i_max_SOLV \quad \text{or} \tag{4.44}$$

$$\max_{I=1,\dots,N} \left| (X^{2I})^{k+1,i} - (X^{2I})^{k+1,i-1} \right| > tol_p_g_max \quad \text{or} \tag{4.45}$$

$$\max_{I=1,\dots,N} \left| (X^{2I})^{k+1,i} - (X^{2I})^{k+1,i-1} \right| > tol_p_g_max\,, \tag{4.46}$$

where i_max_SOLV is the maximum number of iterations, $tol_p_l_max > tol_p_l_SOLV$ is the maximum tolerance for the liquid phase pressure p_l corrections at the nodes and $tol_p_g_max > tol_p_g_SOLV$ is the maximum tolerance for the gas phase pressure p_g corrections at the nodes.

4.5 Time marching scheme

Given a time interval $[t_{ini}, t_{end}]$, the distribution $\boldsymbol{X}(t_{ini})$ at the initial time t_{ini} of liquid phase pressures p_l at the nodes and gas phase pressures p_g at the nodes, the input history during this time interval (megapacker pressures $\sigma(t)$, water mass injection rate $M_w(t)$ and air mass injection rate $M_a(t)$), we want to compute the distribution $\boldsymbol{X}(t_{end})$ at the final time t_{end} of the liquid phase pressures p_l at the nodes and the gas phase pressures p_g at the nodes.

During the integration of the system of differential equations over the time interval $[t_{ini}, t_{end}]$, this interval is divided into time subintervals $[t^i, t^{i+1}]$ ($t^0 = t_{ini}$, $i = 0,1,2,\dots$) of length $\Delta t^i = t^{i+1} - t^i$. Starting from the initial known solution $\boldsymbol{X}(t^0)$, the solutions $\boldsymbol{X}(t^1), \boldsymbol{X}(t^2), \dots$ at times $t^1, t^2, \dots$ are successively computed by solving the systems of algebraic equations associated to the subintervals $[t^0, t^1], [t^1, t^2], \dots$ as explained before. The time marching scheme used is as follows.

(1) **Initial time t^0 and known solution $\boldsymbol{X}(t^0)$**. Set $t^0 = t_{ini}$ and $\boldsymbol{X}(t^0) = \boldsymbol{X}(t_{ini})$.

(2) **Integration from time t^i and known solution $\boldsymbol{X}(t^i)$.** If $t^i = t_{end}$, then the desired solution is $\boldsymbol{X}(t_{end}) = \boldsymbol{X}(t^i)$ and the integration over the time interval $[t_{ini}, t_{end}]$ is finished, otherwise compute a time increment Δt^i (as explained below) and try to solve the corresponding system of algebraic equations.

(3) **Time and known solution update**. If a solution of the system of algebraic equations has been found, then accept the time increment Δt^i and update time $t^i \to t^{i+1} = t^i + \Delta t^i$ and known solution $\boldsymbol{X}(t^i) \to \boldsymbol{X}(t^{i+1})$, otherwise (the solver algorithm has been aborted) reject the time increment Δt^i and keep time $t^i \to t^{i+1} = t^i$ and known solution $\boldsymbol{X}(t^i) \to \boldsymbol{X}(t^{i+1}) = \boldsymbol{X}(t^i)$. Go to step (2).

If the number of rejected time increments at a certain time t^i from a known solution $\boldsymbol{X}(t^i)$ reaches the limit value of 20, the integration over the time interval $[t_{ini}, t_{end}]$ is aborted.

The time increments Δt^i are computed according to the following rules

(1) **Initial time increment (Δt^0)**. It is taken to be d_t_0

(2) **New time increment ($\Delta t^i \rightarrow \Delta t^{i+1}$)**. It is defined by

$$\Delta t^{i+1} = \min(factor \times \Delta t^i, d_t_max, t_{end} - t^i), \tag{4.47}$$

where Δt^i is the previous time increment, $d_t_\max$ is the maximum time increment, t_{end} is the final time and $factor$ is computed as follows:

If the previous system of algebraic equations has been solved using n_iter iterations, then

$$factor = (4/n_iter)^{1/4}. \tag{4.48}$$

This factor is bounded by $\sqrt{2}$ (corresponding to one iteration) and by 0.2 (to avoid too large time step reductions).

If the previous system of algebraic equations could not be solved (the solver algorithm has been aborted), then

$$factor = 1/2. \tag{4.49}$$

Therefore, the time marching scheme used to integrate the system of differential equations over a time interval $[t_{ini}, t_{end}]$ has two parameters, namely the initial time increment d_t_0 and the maximum time increment d_t_max. The limit of 20 consecutive rejected time increments means that it is not accepted to reduce the previous time increment by a factor of $(1/2)^{20} \cong 10^{-6}$.

5 Preliminary results

On the first place, using results of a 2D axisymmetric model of the HG-A test using code_bright, it will be shown that they conform reasonably well to the simplifying assumptions underlying the 1D model. Afterwards, some preliminary results using the 1D model will be presented, including the evolution of the liquid phase pressure, the gas phase pressure, the effective fluid pressure and the evolution of the intrinsic permeability near the test section.

5.1 Verification of the simplifying assumptions

A 2D axisymmetric model for the HG-A test was made using code_bright. Figure 5-1 shows the displacement field, the principal strains field and the liquid phase flow field at $t = 900\ day$, during the water injection tests. The figures show a section near the test section by a half-plane whose boundary coincides with the tunnel axis. The rectangular area on the left corresponds to the test section gravel, the rectangular area on the right corresponds to the Opalinus clay, the rectangular area between the former corresponds to the EDZ. The part of the EDZ in contact with the megapacker (not shown) is about 1.50 m in length, that is, half the length of the megapacker.

The displacement field is such that, on the part of the EDZ in front of the megapacker, is approximately radial, except for a small zone near the test section and. On the Opalinus clay, it is essentially radial. The principal strains field is such that, on the part of the EDZ in front of the megapacker, one principal direction is radially oriented and the principal strain parallel to the tunnel axis vanishes (by axisymmetry, one principal direction must be perpendicular to the tunnel axis), again except for a small zone near the test section. On the Opalinus clay strains nearly vanish. Consequently, except for a small zone near the megapacker ends, the mechanical simplifying assumptions are satisfied: both on the EDZ and on the Opalinus clay, displacements are radial and in plane strain.

The liquid phase flow field is such that, on the part of the EDZ in front of the megapacker, is approximately parallel to the tunnel axis, except for a small zone near the test section. On the Opalinus clay, the liquid flow field vanishes. Consequently, except for a small zone near the megapacker ends, the hydraulic simplifying assumptions in saturated conditions are satisfied: on the EDZ the liquid phase flow is parallel to the tunnel axis and on the Opalinus clay the liquid phase flow vanishes.

Certainly, the results shown are not a verification that the simplifying assumptions on which the 1D model is based hold in the real HG-A test conditions. However, they indicate that it is reasonable to use the 1D simplified model in order to make a first calibration of the model parameters for its use in a 2D axisymmetric model using code_bright.

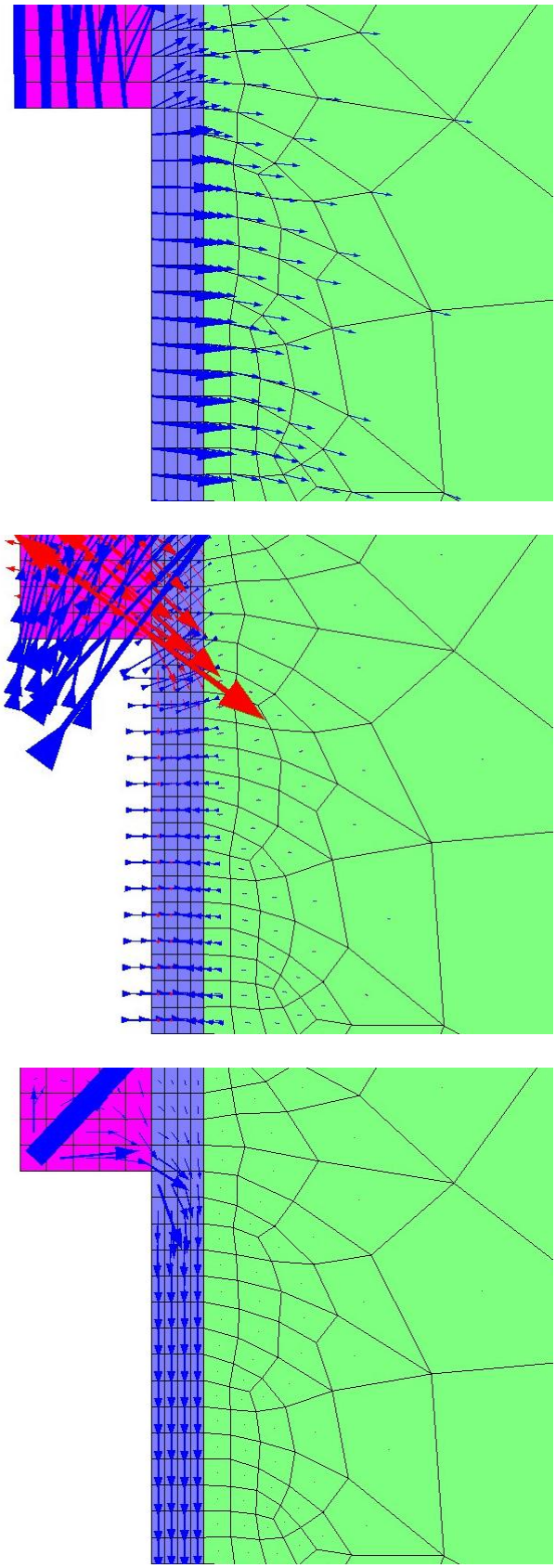

Figure 5-1. 2D axisymmetric code_bright model: (1) displacement field at $t = 900\ day$; (2) principal strains field at $t = 900\ day$; and (3) liquid phase flow field at $t = 900\ day$.

5.2 Predicted evolutions with time of some important variables

The values used for the constitutive parameters both on the Opalinus clay and on the EDZ are summarized in the Table 5-1. The temperature has been assumed to be $T = 20°C$.

The initial ($t = 0$ is 15.04.2006) conditions at the EDZ are $p_l = 0.1MPa$, $p_g = 0.1MPa$ and a uniform initial porosity $\phi^{0EDZ} = 0.12$.

The boundary conditions at the contact between the megapacker and the test section (at $z = 0.00m$) are the polygonal approximations to the records of the sensor M-Flow (liquid phase volume injection rate) and of the sensor M-GasFlow (gas phase volume injection rate) that have been shown in Figure 3-2. In these preliminary calculations, the water exchanges performed prior to the gas injection tests in order to eliminate gas from the test section have not been taken into account.

The boundary conditions at the contact between the megapacker end and the liner (at $z = L$, $L = 3.00m$) are $p_l = 0.1MPa$, $p_g = 0.1MPa$, $\omega_l^a = 1.61\times10^{-5}$ and $\omega_g^w = 1.48\times10^{-2}$, that results from the assumptions $p^{atm} = 0.1MPa$ and $h_r = 100\%$ at the test site.

The evolution of the pressures applied by the megapacker is the polygonal approximation to the record of the sensor PAMegapacker (megapacker pressure) shown in Figure 3-2.

Table 5-1. Values used for the constitutive parameters.

Elastic effective properties					
E'^{OPA}	4.0×10^3	MPa	ν'^{OPA}	0.30	–
E'^{EDZ}	0.8×10^3	MPa	ν'^{EDZ}	0.30	–
Water retention curve					
λ^{EDZ}	0.33	–	P_0^{EDZ}	0.243	MPa
Intrinsic permeability					
k_{matrix}^{EDZ}	1.0×10^{-17}	m^2			
b_0^{EDZ}	1.0×10^{-7}	m	b_{max}^{EDZ}	1.0×10^{-5}	m
a^{EDZ}	1.0×10^{-3}	m	ε_0^{EDZ}	-2.5×10^{-3}	–
Relative permeabilities					
λ_l^{EDZ}	0.33	–			
A_g^{EDZ}	1.00	–	n_g^{EDZ}	3.00	–
Diffusion coefficients					
D_l^{EDZ}	1.4×10^{-4}	m^2/s	Q_l^{EDZ}	2.54×10^{-2}	$MPa\cdot m^3/mol$
D_g^{EDZ}	5.9×10^{-12}	$m^2/s\cdot MPa\cdot°K^{-Q_g}$	Q_g^{EDZ}	2.3	–

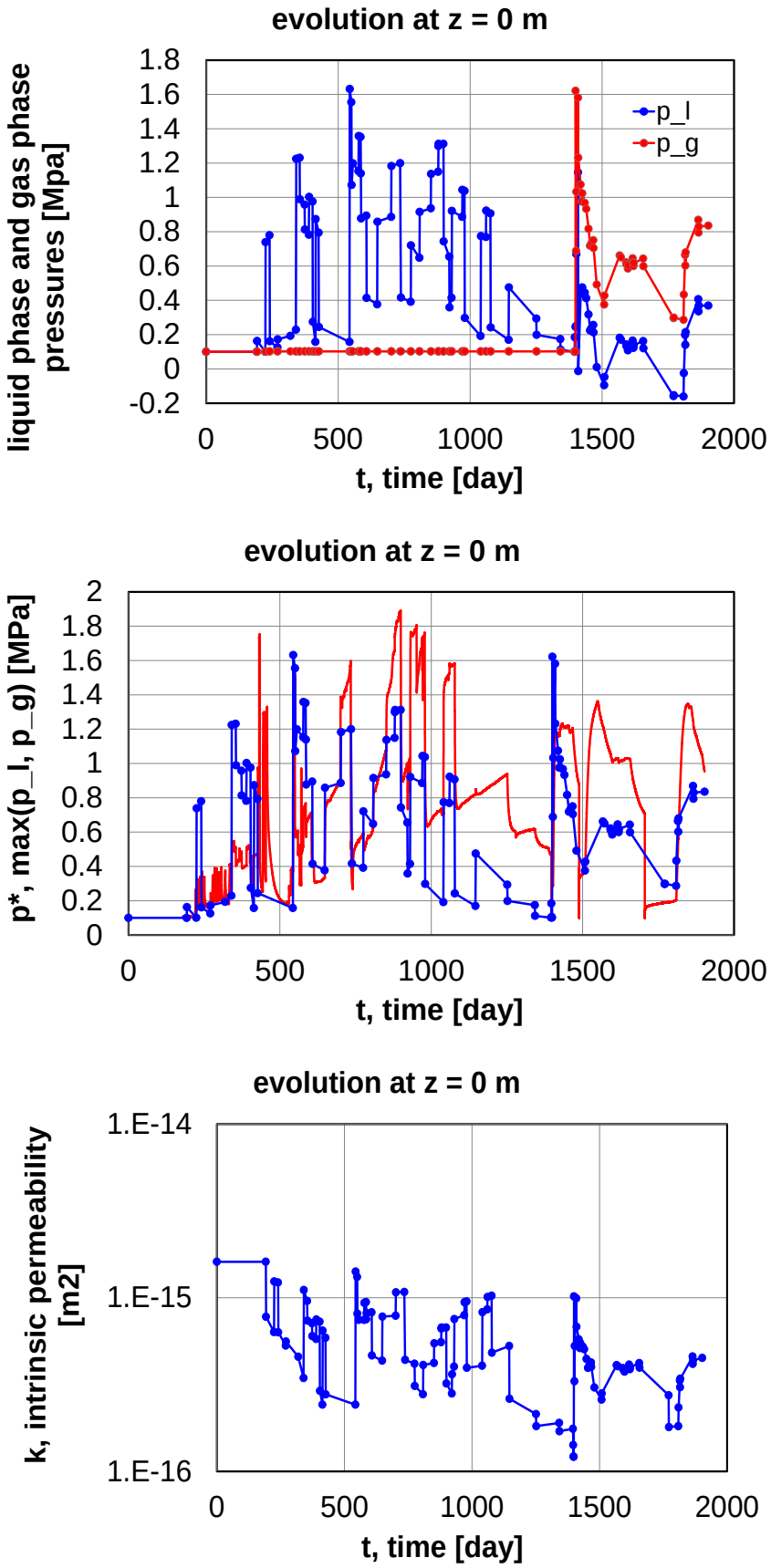

Figure 5-2. 1D model: evolution of (1) liquid phase pressure and gas phase pressure (2) effective fluid pressure; and (3) intrinsic permeability.

The results shown in Figure 5-2 refer to a point located at the contact of the megapacker with the test section (at $z = 0$).

(1) The predicted evolutions of the liquid phase pressure p_l and the gas phase pressure p_g show that, at the considered point, the porous medium remains saturated until around day 1400. Afterwards, due to the gas injection tests, the porous medium desaturates.

(2) The comparison of the predicted evolution of the effective fluid pressure $p*$ compared with the measurements of the sensor M-PEFloor (located on the bottom of the test section) show a reasonable overall agreement, but the sensor shows a tendency of the effective fluid pressure to increase with time which the modelling is not able to reproduce.

(3) The predicted evolution of the intrinsic permeability k shows variations of about one order of magnitude. These predicted large variations of the intrinsic permeability are very important in order to be able to model the HG-A. In fact, the pressure applied by the megapacker on the tunnel wall, tends to close the fractures at the EDZ induced by the tunnel excavation, thereby decreasing the intrinsic permeability.

These results reproduce quite well a number of the features observed in the HG-A test. However, the observed general increasing trend of the fluid pressures is not captured by the model. This might be due to a swelling of the EDZ under the confinement provided by the megapacker that induces a decrease with time of the intrinsic permeability not taken into account by the model.

6 Conclusions

The HG-A test is located at the Mont Terri Underground Laboratory near Saint Ursanne (Jura, Switzerland). It consists of a subhorizontal tunnel of 1.00 in diameter and 13.00 m in length excavated in the ultra-low permeability Opalinus clay. The excavation of the tunnel induced an EDZ (Excavation Damaged Zone) around the tunnel. After the excavation, the last 4.00 m of the tunnel (test section) were backfilled with gravel, a large inflatable packer (megapacker) was installed along the next 3.00 m of the tunnel and a steel liner was installed along the remaining 6.00 m near the mouth of the tunnel in order to guarantee the tunnel stability. After inflating the megapacker, the test section was saturated with water and a series of water injection tests and gas injections tests, whereby water and gas were injected into the test section at varying megapacker pressures, thereby forcing the water and the gas to backflow along the EDZ.

In order to model the flow of liquid and gas through the EDZ, a two track approach has been followed. On the one hand, a 2D axisymmetric numerical model using code_bright (a finite element program developed at the Geotechnical Engineering Department of the UPC) has been prepared. On the other hand, a 1D analytical-numerical model has been developed, as explained in this report, and implemented in an Excel spreadsheet. Both models have essentially the same balance laws, equilibrium restrictions and constitutive relations. They differ in the additional simplifying assumptions assumed in the 1D model and in their numerical solution. The preliminary results obtained using the 1D model are quite similar to those obtained using the 2D axisymmetric model, thereby allowing to use the 1D model to calibrate the model parameters for the 2D axisymmetric model.

The preliminary results obtained using the 1D model reproduce quite well a number of the observed features of the HG-A test. In this regard, the consideration of variations of the intrinsic permeability in the EDZ induced by volumetric deformations is very important. However, the observed trend of the fluid pressures to increase with time is not reproduced by the model. A possible explanation is that a swelling of the EDZ under the confinement provided by the megapacker would induce a decrease of the intrinsic permeability, thereby leading to an increase of the fluid pressures. As this swelling mechanism is not included in the model, this would explain the discrepancies between the predicted and the observed evolution of the fluid pressures.

7 References

[1] Lanyon, G.W., Marschall, P., Trick, T., de La Vaissière, R., Shao, H., Leung, H. (2008) Hydromechanical Evolution and Self-Sealing of Damage Zones around a Microtunnel in a Claystone Formation of the Swiss Jura Mountains. American Rock Mechanics Association ARMA 09-333.

[2] Marschall, P., Distinguin, M., Shao, H., Bossart, P., Enachescu, C., Trick, T. (2006) Creation and Evolution of Damage Zones Around a Microtunnel in a Claystone Formation of the Swiss Jura Mountains. SPE International, SPE-98537-PP.

[3] Marschall, P., Trick, T., Lanyon, G.W., Delay, J., Shao, H. (2008) Hydro-Mechanical Evolution of Damaged Zones around a Microtunnel in a Claystone Formation of the Swiss Jura Mountains. American Rock Mechanics Association ARMA 08-193.

[4] Marschall, P., Horseman, S., Gimmi, T. (2005) Characterisation of Gas Transport Properties of the Opalinus Clay, a Potential Host Rock Formation for Radioactive Waste Disposal. Oil & Gas Science and Technology – Rev. IFP Vol. 60, No. 1, pp. 121-139.

[5] Olivella, S., Carrera, J., Gens, A., Alonso, E.E. (1994) Nonisothermal Multiphase Flow of Brine and Gas through Saline Media. Transport in Porous Media 15, 271-293.

[6] Olivella, S., Alonso, E.E. (2008) Gas flow through clay barriers. Géotechnique 58, No. 3, 157-176.

[7] Trick, T., Marschall, P., Rösli, U., Lettry, Y., Bossart, P., Enachescu, C. (2007) Instrumentation for a gas path through host rock and along sealing experiment. FMGM 2007: Seventh International Simposium on Field Measurements in Geomechanics, ASCE.

APPENDICES

TABLE OF CONTENTS

A1. Some functions and their derivatives with respect to p_l and p_g 46
A1.1. Selected functions of p_l and p_g 47
A1.2. First order derivatives of the selected functions with respect to p_l and p_g 51
A2. Isotropic Linear Elasticity in Plain Strain and Axisymmetry 56
A2.1. Compressible Material ($-1<\nu<1/2$) 57
A2.2. Incompressible Material ($\nu=1/2$) 58
A3. Isotropic Linear Poroelasticity in Plain Strain and Axisymmetry 61
A3.1. Drained conditions with constant p^* 61
A3.2. Undrained conditions 62
A4. 1D Modelling of the HG-A Test 64
A4.1. Solution on planes z = const 64
A4.2. Influence of the effective parameters of the Opalinus clay 67
A5. Weak form of the problem 68
A5.1. Balance of water mass 68
A5.2. Balance of air mass 70
A6. Shape functions, their derivatives and some related integrals 72
A6.1. Shape functions 72
A6.2. Derivatives of the shape functions 72
A6.3. Integrals of the shape functions and of their binary products 73
A7. Nodal Equations 74
A7.1. Balance of water mass 74
A7.2. Balance of air mass 75
A8. Element and node matrices and vectors 76
A8.1. Matrix $\boldsymbol{C}_{[z^I,z^{I+1}]}$ 76
A8.2. Matrix $\boldsymbol{K}_{[z^I,z^{I+1}]}$ 77
A8.3. Vector $\boldsymbol{g}_{[z^I,z^{I+1}]}$ 79
A8.4. Matrix $\boldsymbol{K}_{z^N}$ 79
A8.5. Vector $\boldsymbol{g}_{z^1}$ 79
A8.6. Vector $\boldsymbol{g}_{z^N}$ 79

A1. Some functions and their derivatives with respect to p_l and p_g

It will be convenient to define the real functions $F(s)$ and $G(s,P_0)$ ($P_0 > 0$ and $0 < \lambda < 1$) by

$$F(s) = \exp\left(\frac{-sM_w}{R(273.15+T)\rho_l(T)}\right) \qquad \Rightarrow \qquad 0 < F(s)$$

$$G(s,P_0) = \begin{cases} \left(1+\left(\frac{s}{P_0}\right)^{\frac{1}{1-\lambda}}\right)^{-\lambda} & \text{if } s \geq 0 \\ 1 & \text{if } s < 0 \end{cases} \qquad \Rightarrow \qquad 0 < G(s) \leq 1$$

These functions are smooth (they and their derivatives of all orders are continuous) and have the property that their derivatives may be written using the original functions

$$F'(s) = -\frac{M_w}{R(273.15+T)\rho_l(T)} F(s) \qquad \Rightarrow \qquad 0 < F'(s)$$

$$\frac{\partial G}{\partial s}(s,P_0) = \begin{cases} -\frac{1}{P_0}\frac{\lambda}{1-\lambda} G^{1/\lambda}(s)\left(1-G^{1/\lambda}(s)\right)^{\lambda} & \text{if } s \geq 0 \\ 0 & \text{if } s < 0 \end{cases} \qquad \Rightarrow \qquad \frac{\partial G}{\partial s}(s,P_0) \leq 0$$

$$\frac{\partial G}{\partial P_0}(s,P_0) = \begin{cases} \frac{s}{P_0^{\,2}}\frac{\lambda}{1-\lambda} G^{1/\lambda}(s)\left(1-G^{1/\lambda}(s)\right)^{\lambda} & \text{if } s \geq 0 \\ 0 & \text{if } s < 0 \end{cases} \qquad \Rightarrow \qquad \frac{\partial G}{\partial P_0}(s,P_0) \geq 0$$

A1.1. Selected functions of p_l and p_g

- Partial water and air pressures

From the psychrometric law (2.8), the ideal gas law for dry air (2.37), the constant liquid density $\rho_l = const$ (2.33) and Dalton's law (2.36), we get

$$p_g^w(p_l, p_g) = p_{g0}^w(T)\exp\left(\frac{-(p_g - p_l)M_w}{R(273.15+T)\rho_l(T)}\right) \quad \text{and}$$
$$p_g^a(p_l, p_g) = p_g - p_{g0}^w(T)\exp\left(\frac{-(p_g - p_l)M_w}{R(273.15+T)\rho_l(T)}\right).$$

Using the definition of $F(s)$, we get

$$p_g^w(p_l, p_g) = p_{g0}^w(T)F(p_g - p_l)$$
$$p_g^a(p_l, p_g) = p_g - p_{g0}^w(T)F(p_g - p_l)$$

where, in order to simplify the notation, the dependence on the constant T has been omitted.

- Mass fractions in the liquid phase

Using Henry's law (2.9), the ideal gas law for dry air (2.37), the second of these relations and the constraint $\omega_l^a + \omega_l^w = 1$, we get

$$\omega_l^a(p_l, p_g) = \frac{M_a}{HM_w} p_g^a(p_l, p_g) \quad \text{and}$$
$$\omega_l^w(p_l, p_g) = 1 - \frac{M_a}{HM_w} p_g^a(p_l, p_g).$$

Using the expressions for the functions $p_g^w(p_l, p_g)$ and $p_g^a(p_l, p_g)$, we get

$$\omega_l^a(p_l, p_g) = \frac{M_a}{HM_w}[p_g - p_{g0}^w(T)F(p_g - p_l)]$$
$$\omega_l^w(p_l, p_g) = 1 - \frac{M_a}{HM_w}[p_g - p_{g0}^w(T)F(p_g - p_l)]$$

- Mass fractions in the gas phase

From the ideal gas law for dry air (2.37), the ideal gas law for water vapour (2.38) and the constraint $\omega_g^a + \omega_g^w = 1$, we get

$$\omega_g^w(p_l, p_g) = \frac{M_w p_g^w(p_l, p_g)}{M_w p_g^w(p_l, p_g) + M_a p_g^a(p_l, p_g)} \quad \text{and}$$

$$\omega_g^a(p_l, p_g) = \frac{M_a p_g^a(p_l, p_g)}{M_w p_g^w(p_l, p_g) + M_a p_g^a(p_l, p_g)}.$$

Using the expressions for the functions $p_g^w(p_l, p_g)$ and $p_g^a(p_l, p_g)$, we get

$$\omega_g^w(p_l, p_g) = \left(1 - \frac{M_a}{M_w} + \frac{M_a}{M_w p_{g0}^w(T)} \frac{p_g}{F(p_g - p_l)}\right)^{-1}$$

$$\omega_g^a(p_l, p_g) = 1 - \left(1 - \frac{M_a}{M_w} + \frac{M_a}{M_w p_{g0}^w(T)} \frac{p_g}{F(p_g - p_l)}\right)^{-1}$$

- Density of the gas phase

Using in these relations Dalton's law (2.36), the ideal gas law for dry air (2.37), the ideal gas law for water vapour (2.38), we get

$$\rho_g(p_l, p_g) = \frac{M_w p_g^w(p_l, p_g) + M_a p_g^a(p_l, p_g)}{R(273.15 + T)}$$

Using the expressions for the functions $p_g^w(p_l, p_g)$ and $p_g^a(p_l, p_g)$, we get

$$\rho_g(p_l, p_g) = \frac{M_a p_g + (M_w - M_a) p_{g0}^w(T) F(p_g - p_l)}{R(273.15 + T)}$$

- Effective fluid pressure

According to equation (2.40), the effective fluid pressure reads

$$p^*(p_l, p_g) = \max(p_l, p_g)$$

- Volumetric deformation

According to equation (3.9), the volumetric deformation reads

$$\varepsilon_{vol}(p_l, p_g) = 2[A_{\Delta\sigma}^{EDZ}(\sigma - \sigma_0) + A_{\Delta p^*}^{EDZ}(p^* - p^*_0)]$$

Using the expression for the function $p^*(p_l, p_g)$, we get

$$\varepsilon_{vol}(p_l, p_g) = 2[A_{\Delta\sigma}^{EDZ}(\sigma - \sigma_0) + A_{\Delta p^*}^{EDZ}(\max(p_l, p_g) - p^*_0)]$$

where, σ_0 is the constant initial megapacker pressure, p^*_0 is the constant initial effective fluid pressure and, in order to simplify the notation, the dependence on the megapacker pressure σ has not been explicitly indicated.

- Intrinsic permeability

According to equations (2.22) and (2.23), the intrinsic permeability reads

$$k(p_l, p_g) = \begin{cases} k_{matrix} + \dfrac{1}{12a}\left[b_0\right]^3 & \varepsilon_{vol}(p_l, p_g) \le \varepsilon_{vol0} \\ k_{matrix} + \dfrac{1}{12a}\left[b_0 + (\varepsilon_{vol} - \varepsilon_{vol0})a\right]^3 & \varepsilon_{vol0} \le \varepsilon_{vol}(p_l, p_g) \le \varepsilon_{vol0} + (b_{max} - b_0)/a \\ k_{matrix} + \dfrac{1}{12a}\left[b_{max}\right]^3 & \varepsilon_{vol0} + (b_{max} - b_0)/a \le \varepsilon_{vol}(p_l, p_g) \end{cases}$$

where, in order to simplify the notation, the dependence on the megapacker pressure σ coming from ε_{vol} has not been explicitly indicated.

- Parameter in van Genuchten's water retention curve

According to equations (2.16) and (2.17), the parameter P_0 in van Genuchten's water retention curve reads

$$P_0(p_l, p_g) = P_{0\,matrix}\sqrt[3]{\frac{k_{matrix}}{k(p_l, p_g)}}$$

where, in order to simplify the notation, the dependence on the megapacker pressure σ coming from k has not been explicitly indicated.

- Degree of saturation of the liquid and gas phases

According to equation (2.15) and the constraint $S_l + S_g = 1$, we get

$$S_l(p_l, p_g) = \begin{cases} \left(1 + \left(\dfrac{p_g - p_l}{P_0}\right)^{\frac{1}{1-\lambda}}\right)^{-\lambda} & p_g > p_l \\ 1 & p_g \le p_l \end{cases}$$

$$S_g(p_l, p_g) = \begin{cases} 1 - \left(1 + \left(\dfrac{p_g - p_l}{P_0}\right)^{\frac{1}{1-\lambda}}\right)^{-\lambda} & p_g > p_l \\ 0 & p_g \le p_l \end{cases}.$$

Or, using the definition of $G(s, P_0)$, we get

$$S_l(p_l, p_g) = G(p_g - p_l, P_0)$$
$$S_g(p_l, p_g) = 1 - G(p_g - p_l, P_0)$$

where, in order to simplify the notation, the dependence on the megapacker pressure σ coming from P_0 has not been explicitly indicated.

A1.2. First order derivatives of the selected functions with respect to p_l and p_g

- Partial water and air pressures

$$\frac{\partial p_g^w}{\partial p_l}(p_l, p_g) = \frac{p_{g0}^w(T)}{\rho_l(T)} \frac{M_w}{R(273.15+T)} F(p_g - p_l)$$
$$\frac{\partial p_g^w}{\partial p_g}(p_l, p_g) = -\frac{p_{g0}^w(T)}{\rho_l(T)} \frac{M_w}{R(273.15+T)} F(p_g - p_l)$$

$$\frac{\partial p_g^a}{\partial p_l}(p_l, p_g) = -\frac{p_{g0}^w(T)}{\rho_l(T)} \frac{M_w}{R(273.15+T)} F(p_g - p_l)$$
$$\frac{\partial p_g^a}{\partial p_g}(p_l, p_g) = 1 + \frac{p_{g0}^w(T)}{\rho_l(T)} \frac{M_w}{R(273.15+T)} F(p_g - p_l)$$

- Mass fractions in the liquid phase

$$\frac{\partial \omega_l^a}{\partial p_l}(p_l, p_g) = -\frac{M_a}{HM_w}\left(\frac{p_{g0}^w(T)}{\rho_l(T)} \frac{M_w}{R(273.15+T)} F(p_g - p_l)\right)$$
$$\frac{\partial \omega_l^a}{\partial p_g}(p_l, p_g) = \frac{M_a}{HM_w}\left(1 + \frac{p_{g0}^w(T)}{\rho_l(T)} \frac{M_w}{R(273.15+T)} F(p_g - p_l)\right)$$

$$\frac{\partial \omega_l^w}{\partial p_l}(p_l, p_g) = \frac{M_a}{HM_w}\left(\frac{p_{g0}^w(T)}{\rho_l(T)} \frac{M_w}{R(273.15+T)} F(p_g - p_l)\right)$$
$$\frac{\partial \omega_l^w}{\partial p_g}(p_l, p_g) = -\frac{M_a}{HM_w}\left(1 + \frac{p_{g0}^w(T)}{\rho_l(T)} \frac{M_w}{R(273.15+T)} F(p_g - p_l)\right)$$

- Mass fractions of the gas phase

$$\frac{\partial \omega_g^w}{\partial p_l}(p_l, p_g) = \left(1 - \frac{M_a}{M_w} + \frac{M_a}{M_w p_{g0}^w(T)} \frac{p_g}{F(p_g - p_l)}\right)^{-2} \frac{M_a}{M_w p_{g0}^w(T)} \frac{1}{F(p_g - p_l)}\left(\frac{M_w p_g}{R(273.15+T)\rho_l(T)}\right)$$
$$\frac{\partial \omega_g^w}{\partial p_g}(p_l, p_g) = -\left(1 - \frac{M_a}{M_w} + \frac{M_a}{M_w p_{g0}^w(T)} \frac{p_g}{F(p_g - p_l)}\right)^{-2} \frac{M_a}{M_w p_{g0}^w(T)} \frac{1}{F(p_g - p_l)}\left(1 + \frac{M_w p_g}{R(273.15+T)\rho_l(T)}\right)$$

$$\frac{\partial \omega_g^a}{\partial p_l}(p_l, p_g) = -\left(1 - \frac{M_a}{M_w} + \frac{M_a}{M_w p_{g0}^w(T)} \frac{p_g}{F(p_g - p_l)}\right)^{-2} \frac{M_a}{p_{g0}^w(T)} \frac{1}{F(p_g - p_l)}\left(\frac{M_w p_g}{R(273.15+T)\rho_l(T)}\right)$$
$$\frac{\partial \omega_g^a}{\partial p_g}(p_l, p_g) = \left(1 - \frac{M_a}{M_w} + \frac{M_a}{M_w p_{g0}^w(T)} \frac{p_g}{F(p_g - p_l)}\right)^{-2} \frac{M_a}{M_w p_{g0}^w(T)} \frac{1}{F(p_g - p_l)}\left(1 + \frac{M_w p_g}{R(273.15+T)\rho_l(T)}\right)$$

- Gas phase density

$$\frac{\partial \rho_g}{\partial p_l}(p_l, p_g) = \frac{M_w}{R(273.15+T)}\frac{\partial p_g^w}{\partial p_l} + \frac{M_a}{R(273.15+T)}\frac{\partial p_g^a}{\partial p_l}$$
$$\frac{\partial \rho_g}{\partial p_g}(p_l, p_g) = \frac{M_w}{R(273.15+T)}\frac{\partial p_g^w}{\partial p_g} + \frac{M_a}{R(273.15+T)}\frac{\partial p_g^a}{\partial p_g}$$

Using the expressions for $\partial p_g^w / \partial p_l$, $\partial p_g^a / \partial p_l$, $\partial p_g^w / \partial p_g$ and $\partial p_g^a / \partial p_g$, we get

$$\frac{\partial \rho_g}{\partial p_l}(p_l, p_g) = \frac{p_{g0}^w(T)}{\rho_l(T)}\left(\frac{M_w}{R(273.15+T)}\right)^2 F(p_g - p_l)\left(1 - \frac{M_a}{M_w}\right)$$
$$\frac{\partial \rho_g}{\partial p_g}(p_l, p_g) = \frac{M_a}{R(273.15+T)} - \frac{p_{g0}^w(T)}{\rho_l(T)}\left(\frac{M_w}{R(273.15+T)}\right)^2 F(p_g - p_l)\left(1 - \frac{M_a}{M_w}\right)$$

- Effective fluid pressure

$$\frac{\partial p^*}{\partial p_l}(p_l, p_g) = \begin{cases} 1 & p_l > p_g \\ 0 & p_l < p_g \end{cases}$$
$$\frac{\partial p^*}{\partial p_g}(p_l, p_g) = \begin{cases} 0 & p_l > p_g \\ 1 & p_l < p_g \end{cases}$$

Note that if $p_l = p_g$ these derivatives are not defined.

- Volumetric deformation

$$\frac{\partial \varepsilon_{vol}}{\partial p_l}(p_l, p_g) = 2A_{\Delta p^*}^{EDZ}\frac{\partial p^*}{\partial p_l}$$
$$\frac{\partial \varepsilon_{vol}}{\partial p_g}(p_l, p_g) = 2A_{\Delta p^*}^{EDZ}\frac{\partial p^*}{\partial p_g}$$

Using the expressions for $\partial p^* / \partial p_l$ and $\partial p^* / \partial p_g$, we get

$$\frac{\partial \varepsilon_{vol}}{\partial p_l}(p_l, p_g) = \begin{cases} 2A_{\Delta p^*}^{EDZ} & p_l > p_g \\ 0 & p_l < p_g \end{cases}$$
$$\frac{\partial \varepsilon_{vol}}{\partial p_g}(p_l, p_g) = \begin{cases} 0 & p_l > p_g \\ 2A_{\Delta p^*}^{EDZ} & p_l < p_g \end{cases}$$

Note that if $p_l = p_g$ these derivatives are not defined.

- Intrinsic permeability

$$\frac{dk}{d\varepsilon_{vol}}(\varepsilon_{vol}) = \begin{cases} 0 & \varepsilon_{vol} \le \varepsilon_{vol0} \\ \frac{1}{4}\left[b_0 + (\varepsilon_{vol} - \varepsilon_{vol0})a\right]^2 & \varepsilon_{vol0} \le \varepsilon_{vol} \le \varepsilon_{vol0} + (b_{max} - b_0)/a \\ 0 & \varepsilon_{vol0} + (b_{max} - b_0)/a \le \varepsilon_{vol} \end{cases}$$

Since ε_{vol} depends on p_l and p_g, application of the chain rule yields

$$\frac{\partial k}{\partial p_l}(p_l, p_g) = \frac{dk}{d\varepsilon_{vol}}\frac{\partial \varepsilon_{vol}}{\partial p_l}$$
$$\frac{\partial k}{\partial p_g}(p_l, p_g) = \frac{dk}{d\varepsilon_{vol}}\frac{\partial \varepsilon_{vol}}{\partial p_g}$$

Note that if $p_l = p_g$ these derivatives are not defined.

- Parameter in van Genuchten's water retention curve

$$\frac{dP_0}{dk}(k) = -\frac{1}{3}P_{0\,matrix}\sqrt[3]{\frac{k_{matrix}}{k^4}}$$

Since k depends on ε_{vol} and ε_{vol} depends on p_l and p_g, application of the chain rule yields

$$\frac{\partial P_0}{\partial p_l}(p_l, p_g) = \frac{dP_0}{dk}\frac{dk}{d\varepsilon_{vol}}\frac{\partial \varepsilon_{vol}}{\partial p_l}$$
$$\frac{\partial P_0}{\partial p_g}(p_l, p_g) = \frac{dP_0}{dk}\frac{dk}{d\varepsilon_{vol}}\frac{\partial \varepsilon_{vol}}{\partial p_g}$$

Using the expressions of $\partial\varepsilon_{vol}(p_l, p_g)/\partial p_l$ and $\partial\varepsilon_{vol}(p_l, p_g)/\partial p_g$, we get

$$\frac{\partial P_0}{\partial p_l}(p_l, p_g) = \begin{cases} 2A_{\Delta p*}^{EDZ}\frac{dP_0}{dk}\frac{dk}{d\varepsilon_{vol}} & p_l > p_g \\ 0 & p_l < p_g \end{cases}$$
$$\frac{\partial \varepsilon_{vol}}{\partial p_g}(p_l, p_g) = \begin{cases} 0 & p_l > p_g \\ 2A_{\Delta p*}^{EDZ}\frac{dP_0}{dk}\frac{dk}{d\varepsilon_{vol}} & p_l < p_g \end{cases}$$

Note that if $p_l = p_g$ these derivatives are not defined.

- Degree of saturation of the liquid and gas phases

Since S_l and S_g depend on p_l and p_g via the function $G(s,P_0)$, where $s = p_g - p_l$ and $P_0 = P_0(p_l, p_g)$, application of the chain rule yields

$$\frac{\partial S_l}{\partial p_l}(p_l, p_g) = \frac{\partial G}{\partial s}\frac{\partial s}{\partial p_l} + \frac{\partial G}{\partial P_0}\frac{\partial P_0}{\partial p_l}$$
$$\frac{\partial S_l}{\partial p_g}(p_l, p_g) = \frac{\partial G}{\partial s}\frac{\partial s}{\partial p_g} + \frac{\partial G}{\partial P_0}\frac{\partial P_0}{\partial p_g}$$

$$\frac{\partial S_g}{\partial p_l}(p_l, p_g) = -\left(\frac{\partial G}{\partial s}\frac{\partial s}{\partial p_l} + \frac{\partial G}{\partial P_0}\frac{\partial P_0}{\partial p_l}\right)$$
$$\frac{\partial S_l}{\partial p_g}(p_l, p_g) = -\left(\frac{\partial G}{\partial s}\frac{\partial s}{\partial p_g} + \frac{\partial G}{\partial P_0}\frac{\partial P_0}{\partial p_g}\right)$$

Using the expressions of $\partial G(s,P_0)/\partial s$ and $\partial G(s,P_0)/\partial P_0$ in terms of $G(s,P_0)$, we get

$$\frac{\partial S_l}{\partial p_l}(p_l, p_g) = \begin{cases} \left(\frac{s}{P_0^2}\frac{\partial P_0}{\partial p_l} + \frac{1}{P_0}\right)\frac{\lambda}{1-\lambda} G^{1/\lambda}(s,P_0)\left(1 - G^{1/\lambda}(s,P_0)\right)^{\lambda} & \text{if } p_g > p_l \\ 0 & \text{if } p_g < p_l \end{cases}$$
$$\frac{\partial S_l}{\partial p_g}(p_l, p_g) = \begin{cases} \left(\frac{s}{P_0^2}\frac{\partial P_0}{\partial p_g} - \frac{1}{P_0}\right)\frac{\lambda}{1-\lambda} G^{1/\lambda}(s,P_0)\left(1 - G^{1/\lambda}(s,P_0)\right)^{\lambda} & \text{if } p_g > p_l \\ 0 & \text{if } p_g < p_l \end{cases}$$

Using the expressions of $\partial P_0(p_l, p_g)/\partial p_l$ and $\partial P_0(p_l, p_g)/\partial p_g$, we get

$$\frac{\partial S_l}{\partial p_l}(p_l, p_g) = \begin{cases} \left(\frac{1}{P_0}\right)\frac{\lambda}{1-\lambda} G^{1/\lambda}(s,P_0)\left(1 - G^{1/\lambda}(s,P_0)\right)^{\lambda} & \text{if } p_g > p_l \\ 0 & \text{if } p_g < p_l \end{cases}$$
$$\frac{\partial S_l}{\partial p_g}(p_l, p_g) = \begin{cases} \left(\frac{s}{P_0^2} 2A_{\Delta p*}^{EDZ}\frac{dP_0}{dk}\frac{dk}{d\varepsilon_{vol}} - \frac{1}{P_0}\right)\frac{\lambda}{1-\lambda} G^{1/\lambda}(s,P_0)\left(1 - G^{1/\lambda}(s,P_0)\right)^{\lambda} & \text{if } p_g > p_l \\ 0 & \text{if } p_g < p_l \end{cases}$$

Finally, using relation $S_l + S_g = 1$, we get

$$\frac{\partial S_g}{\partial p_l}(p_l, p_g) = \begin{cases} -\left(\frac{1}{P_0}\right)\frac{\lambda}{1-\lambda} G^{1/\lambda}(s, P_0)\left(1 - G^{1/\lambda}(s, P_0)\right)^{\lambda} & \text{if } p_g > p_l \\ 0 & \text{if } p_g < p_l \end{cases}$$

$$\frac{\partial S_g}{\partial p_g}(p_l, p_g) = \begin{cases} -\left(\frac{s}{P_0^2} 2A_{\Delta p^*}^{EDZ} \frac{dP_0}{dk}\frac{dk}{d\varepsilon_{vol}} - \frac{1}{P_0}\right)\frac{\lambda}{1-\lambda} G^{1/\lambda}(s, P_0)\left(1 - G^{1/\lambda}(s, P_0)\right)^{\lambda} & \text{if } p_g > p_l \\ 0 & \text{if } p_g < p_l \end{cases}$$

Note that if $p_l = p_g$ these derivatives are not defined.

A2. Isotropic Linear Elasticity in Plain Strain and Axisymmetry

Plane strain and axisymmetry of the problem dictate that

(1) The only non-vanishing component of the displacement vector field is u_r, the only non-vanishing components of the strain tensor field are ε_r and ε_θ and the only non-vanishing components of the stress tensor field are σ_r, σ_θ and σ_z. These non-vanishing components are a function only of r.

(2) The strain tensor is computed from the displacement vector according to

$$\varepsilon_r = \frac{du_r}{dr} \quad \text{and} \quad \varepsilon_\theta = \frac{u_r}{r}.$$

(3) The increment with respect to the initial state of the stress tensor is related to the strain tensor by Hooke's law,

$$\varepsilon_r = \frac{1}{E}(\Delta\sigma_r - \nu\Delta\sigma_\theta - \nu\Delta\sigma_z)$$
$$\varepsilon_\theta = \frac{1}{E}(-\nu\Delta\sigma_r + \Delta\sigma_\theta - \nu\Delta\sigma_z)$$
$$\varepsilon_z = \frac{1}{E}(-\nu\Delta\sigma_r - \nu\Delta\sigma_\theta + \Delta\sigma_z),$$

where E is Young's modulus and ν is Poisson's ratio. These linear relations between ε_r, ε_θ, ε_z and $\Delta\sigma_r$, $\Delta\sigma_\theta$, $\Delta\sigma_z$ may be inverted if and only if its determinant does not vanish

$$\frac{-1}{E^3}(2\nu^3 + 3\nu^2 - 1) = \frac{-1}{E^3}(2\nu - 1)(\nu + 1)^2 \neq 0.$$

The case $\nu = 1/2$ corresponds to an infinite bulk modulus $K = E/[3(1-2\nu)]$ (material incompressibility) and the case $\nu = -1$ corresponds to an infinite shear modulus $G = E/[2(1+\nu)]$.

(4) The balance of momentum in the absence of body forces reads

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0.$$

Two possibilities arise, namely, the material is compressible ($\nu \neq 1/2$) and the material is incompressible ($\nu = 1/2$).

A2.1. Compressible Material ($-1<\nu<1/2$)

In this case, it is possible to invert Hooke's law.

$$\Delta\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_r + \nu\varepsilon_\theta + \nu\varepsilon_z]$$
$$\Delta\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_r + (1-\nu)\varepsilon_\theta + \nu\varepsilon_z]$$
$$\Delta\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_r + \nu\varepsilon_\theta + (1-\nu)\varepsilon_z]$$

Using these relations, the expressions of the non-vanishing components of the strain tensor and the plane strain condition $\varepsilon_z = 0$, we get

$$\frac{d(\Delta\sigma_r)}{dr} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{d\varepsilon_r}{dr} + \nu\frac{d\varepsilon_\theta}{dr}\right] = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{d^2u_r}{dr^2} + \frac{\nu}{r}\frac{du_r}{dr} - \frac{\nu}{r^2}u_r\right]$$

$$\Delta(\sigma_r - \sigma_\theta) = \frac{E}{(1+\nu)(1-2\nu)}[(1-2\nu)(\varepsilon_r - \varepsilon_\theta)] = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\frac{du_r}{dr} - (1-2\nu)\frac{u_r}{r}\right].$$

Substituting them in the balance of momentum, we get

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2}\right) = \frac{d\sigma_r^0}{dr} + \frac{\sigma_r^0 - \sigma_\theta^0}{r}$$

And, since the initial stress state is in equilibrium and $E>0$, we get the equation

$$r^2\frac{d^2u_r}{dr^2} + r\frac{du_r}{dr} - u_r = 0,$$

whose general solution is

$$u_r(r) = Ar + \frac{B}{r},$$

where A and B are constants. Therefore, the non-vanishing components of the strain tensor read

$$\varepsilon_r(r) = A - \frac{B}{r^2} \quad \text{and}$$
$$\varepsilon_\theta(r) = A + \frac{B}{r^2},$$

and the non-vanishing components of the increment of the stress tensor read

$$\Delta\sigma_r(r) = \frac{E}{(1+\nu)(1-2\nu)}\left[A-(1-2\nu)\frac{B}{r^2}\right],$$
$$\Delta\sigma_\theta(r) = \frac{E}{(1+\nu)(1-2\nu)}\left[A+(1-2\nu)\frac{B}{r^2}\right] \quad \text{and}$$
$$\Delta\sigma_z(r) = \frac{E}{(1+\nu)(1-2\nu)}\left[2\nu A\right].$$

Therefore, for $-1<\nu<1/2$, the non-vanishing components of the displacement vector, the strain tensor and the stress tensor are

$$u_r(r) = Ar + \frac{B}{r},$$
$$\varepsilon_r(r) = A - \frac{B}{r^2}$$
$$\varepsilon_\theta(r) = A + \frac{B}{r^2}$$
$$\sigma_r(r) = \frac{E}{(1+\nu)(1-2\nu)}A - \frac{E}{(1+\nu)}\frac{B}{r^2} + \sigma_r^0(r),$$
$$\sigma_\theta(r) = \frac{E}{(1+\nu)(1-2\nu)}A + \frac{E}{(1+\nu)}\frac{B}{r^2} + \sigma_\theta^0(r) \quad \text{and}$$
$$\sigma_z(r) = \frac{E}{(1+\nu)(1-2\nu)}\left[2\nu A\right] + \sigma_z^0(r),$$

where A and B are constants. Note that the volumetric deformation $\varepsilon_{vol} = \varepsilon_r + \varepsilon_\theta + \varepsilon_z$ does not depend on r

$$\varepsilon_{vol}(r) = 2A.$$

A2.2. Incompressible Material ($\nu = 1/2$)

Hooke's law with $\nu = 1/2$ reads

$$\varepsilon_r = \frac{1}{2E}[2\Delta\sigma_r - \Delta\sigma_\theta - \Delta\sigma_z)]$$
$$\varepsilon_\theta = \frac{1}{2E}[-\Delta\sigma_r + 2\Delta\sigma_\theta - \Delta\sigma_z)]$$
$$\varepsilon_z = \frac{1}{2E}[-\Delta\sigma_r - \Delta\sigma_\theta + 2\Delta\sigma_z)]$$

Adding these equations, we get the incompressibility condition

$$\varepsilon_r + \varepsilon_\theta + \varepsilon_z = 0$$

Using the plane strain condition $\varepsilon_z = 0$ and the expressions of the non-vanishing components of the strain tensor, we get the equation

$$\frac{du_r}{dr} + \frac{u_r}{r} = 0,$$

whose general solution is

$$u_r(r) = \frac{B}{r},$$

where B is a constant. Therefore, the non-vanishing components of the strain tensor read

$$\varepsilon_r(r) = -\frac{B}{r^2}$$
$$\varepsilon_\theta(r) = \frac{B}{r^2}$$

Since in this case Hooke's law is not invertible, we will use Hooke's law and the balance of momentum in order to determine the non-vanishing components of the stress tensor.

Using the plane strain condition $\varepsilon_z = 0$ and the third equation in Hooke's law, we have

$$\Delta\sigma_z = \frac{1}{2}(\Delta\sigma_r + \Delta\sigma_\theta)$$

Substituting this expression in the first equation in Hooke's law, we get

$$\varepsilon_r = \frac{3}{4E}(\Delta\sigma_r - \Delta\sigma_\theta),$$

which, taking into account the expression of ε_r derived above, yields

$$\Delta\sigma_r - \Delta\sigma_\theta = -\frac{4EB}{3r^2}$$

Using this result in the balance of momentum, we have that

$$0 = \frac{d\Delta\sigma_r}{dr} + \frac{\Delta\sigma_r - \Delta\sigma_\theta}{r} - \left(\frac{d\sigma_r^0}{dr} + \frac{\sigma_r^0 - \sigma_\theta^0}{r}\right) = \frac{d\Delta\sigma_r}{dr} - \frac{4EB}{3r^2} - \left(\frac{d\sigma_r^0}{dr} + \frac{\sigma_r^0 - \sigma_\theta^0}{r}\right)$$

And, since the initial stress state is in equilibrium, we get the equation

$$\frac{d\Delta\sigma_r}{dr} - \frac{4EB}{3r^2} = 0,$$

whose general solution is

$$\Delta\sigma_r(r) = -\frac{2EB}{3r^2} + C\,,$$

where C is a constant. Using the relationship between $\Delta\sigma_r$ and $\Delta\sigma_\theta$ found above, we get

$$\Delta\sigma_\theta(r) = \Delta\sigma_r(r) + \frac{4EB}{3r^2} = \frac{2EB}{3r^2} + C\,.$$

Finally, using the expression of $\Delta\sigma_r$ in terms of $\Delta\sigma_r$ and $\Delta\sigma_\theta$ found above, we get

$$\Delta\sigma_z = \frac{1}{2}(\Delta\sigma_r + \Delta\sigma_\theta) = C\,.$$

Therefore, the non-vanishing components of the increment of the stress tensor read

$$\Delta\sigma_r(r) = -\frac{2EB}{3r^2} + C\,,$$
$$\Delta\sigma_\theta(r) = \frac{2EB}{3r^2} + C\,,$$
$$\Delta\sigma_z(r) = C\,.$$

Therefore, for $\nu = 1/2$, the non-vanishing components of the displacement vector, the strain tensor and the stress tensor are

$$u_r(r) = \frac{B}{r}\,,$$
$$\varepsilon_r(r) = -\frac{B}{r^2}$$
$$\varepsilon_\theta(r) = \frac{B}{r^2}$$
$$\sigma_r(r) = -\frac{E}{(1+\nu)}\frac{B}{r^2} + C + \sigma_r^0(r)\,,$$
$$\sigma_\theta(r) = \frac{E}{(1+\nu)}\frac{B}{r^2} + C + \sigma_\theta^0(r) \quad \text{and}$$
$$\sigma_z(r) = C + \sigma_z^0(r)\,,$$

where B and C are constants. Note that the dependence on the parameters E and ν is via the shear modulus $G = E/[2(1+\nu)]$.

It is worth mentioning that, in general, the equations giving the displacement, strain and stress fields for $\nu = 1/2$ are not the limit as $\nu \to 1/2$ of the corresponding ones for $-1 < \nu < 1/2$.

A3. Isotropic Linear Poroelasticity in Plain Strain and Axisymmetry

We will consider a porous material made of a solid skeleton, a liquid phase and a gas phase, whereby the liquid phase and the gas phase fill the pores of the solid skeleton. For this porous material, a generalized form of Terzaghi's effective stress principle will be accepted, whereby the effective stress tensor σ'_{ij} will be defined by the relation

$$\sigma'_{ij} = \sigma_{ij} + p^* \delta_{ij} \qquad \text{with} \qquad p^* = \max(p_l, p_g)\,,$$

where σ_{ij} is the (total) stress tensor of the porous material (solid skeleton, liquid phase and gas phase) , p_l is the liquid phase pressure and p_l is the gas phase pressure. We will refer to p^* as the (effective) pore pressure.

A3.1. Drained conditions with constant p^*

For an isotropic linear poroelastic material, in plane strain and axisymmetric conditions Hooke's law reads

$$\varepsilon_r = \frac{1}{E'}(\Delta\sigma'_r - \nu'\Delta\sigma'_\theta - \nu'\Delta\sigma'_z)$$
$$\varepsilon_\theta = \frac{1}{E'}(-\nu'\Delta\sigma'_r + \Delta\sigma'_\theta - \nu'\Delta\sigma'_z)$$
$$\varepsilon_z = \frac{1}{E'}(-\nu'\Delta\sigma'_r - \nu'\Delta\sigma'_\theta + \Delta\sigma'_z)\,,$$

where E' is the effective Young's modulus and ν' is the effective Poisson's ratio. We will assume that $-1 < \nu' < 1/2$, so that Hooke's law is invertible.

From the definition of Terzaghi's effective stress, $\Delta\sigma'_r = \Delta\sigma_r + \Delta p^*$, $\Delta\sigma'_\theta = \Delta\sigma_\theta + \Delta p^*$ and $\Delta\sigma'_z = \Delta\sigma_z + \Delta p^*$. Using these relations, Hooke's law reads

$$\varepsilon_r = \frac{1}{E'}(\Delta\sigma_r - \nu'\Delta\sigma_\theta - \nu'\Delta\sigma_z) + \frac{1-2\nu'}{E'}\Delta p^*$$
$$\varepsilon_\theta = \frac{1}{E'}(-\nu'\Delta\sigma_r + \Delta\sigma_\theta - \nu'\Delta\sigma_z) + \frac{1-2\nu'}{E'}\Delta p^*$$
$$\varepsilon_z = \frac{1}{E'}(-\nu'\Delta\sigma_r - \nu'\Delta\sigma_\theta + \Delta\sigma_z) + \frac{1-2\nu'}{E'}\Delta p^*.$$

Thus, in order to determine the non-vanishing components of the strain tensor ε_r, ε_θ and ε_z, the variations of both the pore pressure p^* and the non-vanishing components of the effective stress tensor σ'_r, σ'_θ and σ'_z are needed.

If p^* is constant, then $div\,\boldsymbol{\sigma} = div\,\boldsymbol{\sigma}'$ and we can replace the total stress tensor $\boldsymbol{\sigma}$ by the effective stress tensor $\boldsymbol{\sigma}' = \boldsymbol{\sigma} + p^*\boldsymbol{I}$ in the balance of momentum equation. Therefore, assuming $-1 < \nu' < 1/2$, we can use the results of the previous appendix to find the non-

vanishing components of the displacement vector, the strain tensor and the effective stress tensor

$$u_r(r) = Ar + \frac{B}{r},$$
$$\varepsilon_r(r) = A - \frac{B}{r^2}$$
$$\varepsilon_\theta(r) = A + \frac{B}{r^2}$$
$$\sigma'_r(r) = \frac{E'}{(1+\nu')(1-2\nu')}A - \frac{E'}{(1+\nu')}\frac{B}{r^2} + \sigma'^0_r(r),$$
$$\sigma'_\theta(r) = \frac{E'}{(1+\nu')(1-2\nu')}A + \frac{E'}{(1+\nu')}\frac{B}{r^2} + \sigma'^0_\theta(r) \quad \text{and}$$
$$\sigma'_z(r) = \frac{E'}{(1+\nu')(1-2\nu')}\left[2\nu A\right] + \sigma'^0_z(r),$$

where A and B are constants. It is important to note that the volumetric deformation $\varepsilon_{vol} = \varepsilon_r + \varepsilon_\theta + \varepsilon_z$ does not depend on r

$$\varepsilon_{vol}(r) = 2A.$$

A3.2. Undrained conditions

If the pores of the solid skeleton are filled only with the liquid phase (saturated conditions) and this liquid phase cannot flow, then we are in undrained conditions. If we assume that both the solid skeleton particles and the liquid phase are incompressible, then the poroelastic material cannot have volumetric deformations. Consequently, $\varepsilon_{vol} = \varepsilon_r + \varepsilon_\theta + \varepsilon_z = 0$ and, Hooke's law requires that $\Delta\sigma'_r + \Delta\sigma'_\theta + \Delta\sigma'_z = 0$ which, using the definition of the effective stress ($\Delta\sigma'_r = \Delta\sigma_r + \Delta p^*$, $\Delta\sigma'_\theta = \Delta\sigma_\theta + \Delta p^*$ and $\Delta\sigma'_z = \Delta\sigma_z + \Delta p^*$), is equivalent to

$$\Delta p^* = -\frac{1}{3}(\Delta\sigma_r + \Delta\sigma_\theta + \Delta\sigma_z).$$

Using this relation in Hooke's law, we get

$$\varepsilon_r = \frac{1}{E_u}(\Delta\sigma_r - \nu_u\Delta\sigma_\theta - \nu_u\Delta\sigma_z)$$
$$\varepsilon_\theta = \frac{1}{E_u}(-\nu_u\Delta\sigma_r + \Delta\sigma_\theta - \nu_u\Delta\sigma_z)$$
$$\varepsilon_\theta = \frac{1}{E_u}(-\nu_u\Delta\sigma_r - \nu_u\Delta\sigma_\theta + \Delta\sigma_z),$$

where E_u is the undrained Young's modulus and ν_u is the undrained Poisson's ratio, and are given in terms of E' and ν' by the relations

$$E_u = \frac{3E'}{2(1+\nu')} \quad \text{and} \quad \nu_u = \frac{1}{2}.$$

Thus, in undrained conditions, in order to determine the non-vanishing components of the strain tensor ε_r, ε_θ and ε_z, only the variations of the total stress tensor σ_r, σ_θ and σ_z are needed. The variation of the effective pressure $p*$ is given above. Note that $p* = p_l$, since in saturated conditions $p_l > p_g$.

Since $\nu'_u = 1/2$, we can use the results of the previous appendix to find the non-vanishing components of the displacement vector, the strain tensor, the total stress tensor and the effective pressure

$$u_r(r) = \frac{B}{r},$$
$$\varepsilon_r(r) = -\frac{B}{r^2}$$
$$\varepsilon_\theta(r) = \frac{B}{r^2}$$
$$\sigma_r(r) = -\frac{E'}{(1+\nu')}\frac{B}{r^2} + C + \sigma_r^0(r),$$
$$\sigma_\theta(r) = \frac{E'}{(1+\nu')}\frac{B}{r^2} + C + \sigma_\theta^0(r) \quad \text{and}$$
$$\sigma_z(r) = C + \sigma_z^0(r),$$
$$p*(r) = -C + p*_0(r)$$

where B and C are constants.

A4. 1D Modelling of the HG-A Test

Due to its very low permeability, it will be assumed that the Opalinus clay has an undrained behaviour. Since the EDZ has a much larger permeability, its contact with both the Opalinus clay and the megapacker is impervious.

Because of the assumed axisymmetry, the zone occupied by the EDZ is $r_i < r < r_e$ and the zone occupied by the Opalinus clay is $r_2 < r$. We assume that the effective parameters of the EDZ satisfy $0 < E'^{EDZ}$ and $-1 < \nu'^{EDZ} < 1/2$, whereas the effective parameters of the Opalinus clay satisfy $0 < E'^{OPA}$ and $-1 < \nu'^{OPA} < 1/2$.

The assumed simplifying assumptions imply each slice $z = const$ moves independently and in plane strain and that $p*$ does not depend on r. Therefore, on each slice $z = const$ we may use the results obtained in the previous appendix.

A4.1. Solution on planes z = const

On the Opalinus clay ($r_e < r$), as shown in the previous appendix, the non-vanishing components of the displacement vector, strain tensor, total stress tensor and pore water pressure are given by

$$u_r^{OPA}(r) = \frac{B^{OPA}}{r}$$

$$u_r^{OPA}(r) = \frac{B^{OPA}}{r},$$

$$\varepsilon_r^{OPA}(r) = -\frac{B^{OPA}}{r^2}$$

$$\varepsilon_\theta^{OPA}(r) = \frac{B^{OPA}}{r^2}$$

$$\Delta\sigma_r^{OPA}(r) = -\frac{E'^{OPA}}{(1+\nu'^{OPA})}\frac{B^{OPA}}{r^2} + C^{OPA},$$

$$\Delta\sigma_\theta^{OPA}(r) = \frac{E'^{OPA}}{(1+\nu'^{OPA})}\frac{B^{OPA}}{r^2} + C^{OPA},$$

$$\Delta\sigma_z^{OPA}(r) = C^{OPA} \quad \text{and}$$

$$\Delta p*(r) = -C^{OPA}.$$

On the EDZ ($r_i < r < r_e$) and provided that $p*$ does not depend on r, as shown in the previous appendix, the non-vanishing components of the displacement vector, strain tensor and total stress tensor and are given by

$$u_r^{EDZ}(r) = A^{EDZ} r + \frac{B^{EDZ}}{r}$$

$$\varepsilon_r^{EDZ}(r) = A^{EDZ} - \frac{B^{EDZ}}{r^2}$$

$$\varepsilon_\theta^{EDZ}(r) = A^{EDZ} + \frac{B^{EDZ}}{r^2}$$
$$\Delta\sigma_r^{EDZ}(r) = \frac{E'^{EDZ}}{(1+\nu'^{EDZ})(1-2\nu'^{EDZ})} A^{EDZ} - \frac{E'^{EDZ}}{(1+\nu'^{EDZ})} \frac{B^{EDZ}}{r^2} - \Delta p^*,$$
$$\Delta\sigma_\theta^{EDZ}(r) = \frac{E'^{EDZ}}{(1+\nu'^{EDZ})(1-2\nu'^{EDZ})} A^{EDZ} + \frac{E'^{EDZ}}{(1+\nu'^{EDZ})} \frac{B^{EDZ}}{r^2} - \Delta p^* \quad \text{and}$$
$$\Delta\sigma_z^{EDZ}(r) = \frac{E'^{EDZ}}{(1+\nu'^{EDZ})(1-2\nu'^{EDZ})} [2\nu'^{EDZ} A^{EDZ}] - \Delta p^*,$$

In order to determine the unknown constants B^{OPA}, C^{OPA}, A^{EDZ} and B^{EDZ}, we impose the following conditions

$$\Delta\sigma_r^{OPA}(\infty) = 0,$$
$$\Delta\sigma_r^{OPA}(r_e) = \Delta\sigma_r^{EDZ}(r_e),$$
$$u_r^{OPA}(r_e) = u_r^{EDZ}(r_e) \quad \text{and}$$
$$\Delta\sigma_r^{EDZ}(r_i) = \Delta\sigma^{MP}.$$

Using the expressions of $u_r^{OPA}(r)$, $\Delta\sigma_r^{OPA}(r)$, $u_r^{OPA}(r)$ and $\Delta\sigma_r^{EDZ}(r)$, these conditions read

$$C^{OPA} = 0,$$
$$-\frac{2E_u^{OPA}B^{OPA}}{3r_e^2} + C^{OPA} = \frac{E'^{EDZ}}{(1+\nu'^{EDZ})(1-2\nu'^{EDZ})}\left[A^{EDZ} - (1-2\nu'^{EDZ})\frac{B^{EDZ}}{r_e^2}\right] - \Delta p^*,$$
$$\frac{B^{OPA}}{r_e} = A^{EDZ} r_e + \frac{B^{EDZ}}{r_e} \quad \text{and}$$
$$\frac{E'^{EDZ}}{(1+\nu'^{EDZ})(1-2\nu'^{EDZ})}\left[A^{EDZ} - (1-2\nu'^{EDZ})\frac{B^{EDZ}}{r_i^2}\right] - \Delta p^* = \Delta\sigma^{MP}.$$

Since $C^{OPA} = 0$, this is a linear system of equations in the unknowns B^{OPA}, A^{EDZ} and B^{EDZ}. The determinant of the matrix of the coefficients of the unknowns reads

$$-\frac{1}{r_i^2 r_e^2}\frac{E'^{EDZ}}{(1+\nu'^{EDZ})(1-2\nu'^{EDZ})}\left[\left(1-2\nu'^{EDZ}+\frac{r_i^2}{r_e^2}\right)\frac{E'^{OPA}}{(1+\nu'^{OPA})} + \left(1-\frac{r_i^2}{r_e^2}\right)\frac{E'^{EDZ}}{(1+\nu'^{EDZ})}\right].$$

Since $r_i < r_e$ and it has been assumed that $0 < E'^{EDZ}$, $-1 < \nu'^{EDZ} < 1/2$, $0 < E'^{OPA}$ and $-1 < \nu'^{OPA} < 1/2$, this determinant never vanishes so that this system of equations always has a unique solution. The solution is given by

$$A^{EDZ} = A_{\Delta\sigma}^{EDZ}\Delta\sigma + A_{\Delta p^*}^{EDZ}\Delta p^*,$$
$$B^{EDZ} = B_{\Delta\sigma}^{EDZ}\Delta\sigma + B_{\Delta p^*}^{EDZ}\Delta p^* \quad \text{and}$$
$$B^{OPA} = B_{\Delta\sigma}^{OPA}\Delta\sigma + B_{\Delta p^*}^{OPA}\Delta p^*,$$

where $A_{\Delta\sigma}^{EDZ}$, $A_{\Delta p_w}^{EDZ}$, $B_{\Delta\sigma}^{EDZ}$, $B_{\Delta p_w}^{EDZ}$, $B_{\Delta\sigma}^{OPA}$ and $B_{\Delta p_w}^{OPA}$ are constants depending on the effective properties of the EDZ E'^{EDZ} and ν'^{EDZ}, the effective properties of the Opalinus clay E'^{OPA} and ν'^{OPA} and the geometry of the problem r_i and r_e.

$$A_{\Delta\sigma}^{EDZ} = \frac{(1+\nu'^{EDZ})(1-2\nu'^{EDZ})}{E'^{EDZ}} \left[\frac{\left(\frac{(1+\nu'^{EDZ})}{E'^{EDZ}} \frac{E'^{OPA}}{(1+\nu'^{OPA})} - 1 \right) \frac{r_i^2}{r_e^2}}{\left(1-\frac{r_i^2}{r_e^2}\right) + \frac{(1+\nu'^{EDZ})}{E'^{EDZ}} \frac{E'^{OPA}}{(1+\nu'^{OPA})} \left((1-2\nu'^{EDZ}) + \frac{r_i^2}{r_e^2} \right)} \right]$$

$$A_{\Delta p*}^{EDZ} = \frac{(1+\nu'^{EDZ})(1-2\nu'^{EDZ})}{E'^{EDZ}} \left[\frac{\left(1-\frac{r_i^2}{r_e^2}\right) + \left(\frac{(1+\nu'^{EDZ})}{E'^{EDZ}} \frac{E'^{OPA}}{(1+\nu'^{OPA})} \right) \frac{r_i^2}{r_e^2}}{\left(1-\frac{r_i^2}{r_e^2}\right) + \frac{(1+\nu'^{EDZ})}{E'^{EDZ}} \frac{E'^{OPA}}{(1+\nu'^{OPA})} \left((1-2\nu'^{EDZ}) + \frac{r_i^2}{r_e^2} \right)} \right]$$

$$B_{\Delta\sigma}^{EDZ} = r_i^2 \frac{(1+\nu'^{EDZ})}{E'^{EDZ}} \left[\frac{-1 - \frac{(1+\nu'^{EDZ})}{E'^{EDZ}} \frac{E'^{OPA}}{(1+\nu'^{OPA})} (1-2\nu'^{EDZ})}{\left(1-\frac{r_i^2}{r_e^2}\right) + \frac{(1+\nu'^{EDZ})}{E'^{EDZ}} \frac{E'^{OPA}}{(1+\nu'^{OPA})} \left((1-2\nu'^{EDZ}) + \frac{r_i^2}{r_e^2} \right)} \right]$$

$$B_{\Delta p*}^{EDZ} = r_i^2 \frac{(1+\nu'^{EDZ})}{E'^{EDZ}} \left[\frac{-\frac{(1+\nu'^{EDZ})}{E'^{EDZ}} \frac{E'^{OPA}}{(1+\nu'^{OPA})} (1-2\nu'^{EDZ})}{\left(1-\frac{r_i^2}{r_e^2}\right) + \frac{(1+\nu'^{EDZ})}{E'^{EDZ}} \frac{E'^{OPA}}{(1+\nu'^{OPA})} \left((1-2\nu'^{EDZ}) + \frac{r_i^2}{r_e^2} \right)} \right]$$

$$B_{\Delta\sigma}^{OPA} = r_i^2 \frac{(1+\nu'^{EDZ})}{E'^{EDZ}} \left[\frac{-2(1-\nu'^{EDZ})}{\left(1-\frac{r_i^2}{r_e^2}\right) + \frac{(1+\nu'^{EDZ})}{E'^{EDZ}} \frac{E'^{OPA}}{(1+\nu'^{OPA})} \left((1-2\nu'^{EDZ}) + \frac{r_i^2}{r_e^2} \right)} \right]$$

$$B_{\Delta p*}^{OPA} = r_i^2 \frac{(1+\nu'^{EDZ})}{E'^{EDZ}} \left[\frac{-(1-2\nu'^{EDZ}) \left(1-\frac{r_e^2}{r_i^2}\right)}{\left(1-\frac{r_i^2}{r_e^2}\right) + \frac{(1+\nu'^{EDZ})}{E'^{EDZ}} \frac{E'^{OPA}}{(1+\nu'^{OPA})} \left((1-2\nu'^{EDZ}) + \frac{r_i^2}{r_e^2} \right)} \right]$$

Using these equations, the volumetric deformation in the EDZ is given by

$$\varepsilon_{vol}^{EDZ}(r) = \varepsilon_r^{EDZ}(r) + \varepsilon_\theta^{EDZ}(r) = 2(A_{\Delta\sigma}^{EDZ}\Delta\sigma + A_{\Delta p*}^{EDZ}\Delta p*),$$

hence it does not depend on r.

A4.2. Influence of the effective parameters of the Opalinus clay

Using the former equations, we see that the dependence of the displacement vector, strain tensor and stress tensor fields on the effective parameters of the Opalinus clay E'^{OPA} and ν'^{OPA} is via the shear modulus $G^{OPA} = E'^{OPA}/[2(1+\nu'^{OPA})]$.

Note that at the boundary between the EDZ and the Opalinus clay, we have

$$\Delta\sigma_r^{OPA}(r_e) = -\frac{E'^{OPA}}{(1+\nu'^{OPA})}\frac{1}{r_e}u_r^{OPA}(r_e).$$

Therefore, for given values of r_e and $\Delta\sigma_r^{OPA}(r_e)$, the displacement at the boundary between the EDZ and the Opalinus clay decreases as the shear modulus of the Opalinus clay $G^{OPA} = E'^{OPA}/[2(1+\nu'^{OPA})]$ increases. In the limit as $G^{OPA} = E'^{OPA}/[2(1+\nu'^{OPA})] \to \infty$, we have

$$A_{\Delta\sigma}^{EDZ} = A_{\Delta p*}^{EDZ} = \frac{(1+\nu'^{EDZ})(1-2\nu'^{EDZ})}{E'^{EDZ}}\left[\frac{\frac{r_i^2}{r_e^2}}{1-2\nu'^{EDZ}+\frac{r_i^2}{r_e^2}}\right],$$

$$B_{\Delta\sigma}^{EDZ} = B_{\Delta p*}^{EDZ} = -r_i^2\frac{(1+\nu'^{EDZ})}{E'^{EDZ}}\left[\frac{1-2\nu'^{EDZ}}{1-2\nu'^{EDZ}+\frac{r_i^2}{r_e^2}}\right] \text{ and}$$

$$B_{\Delta\sigma}^{OPA} = B_{\Delta p*}^{OPA} = 0,$$

which imply the relations

$$A^{EDZ} = A_{\Delta\sigma}^{EDZ}(\Delta\sigma+\Delta p*) = A_{\Delta p*}^{EDZ}(\Delta\sigma+\Delta p*),$$
$$B^{EDZ} = B_{\Delta\sigma}^{EDZ}(\Delta\sigma+\Delta p*) = B_{\Delta p*}^{EDZ}(\Delta\sigma+\Delta p*) \quad \text{and}$$
$$B^{OPA} = B_{\Delta\sigma}^{OPA}(\Delta\sigma+\Delta p*) = B_{\Delta p*}^{OPA}(\Delta\sigma+\Delta p*).$$

Therefore, the Opalinus clay behaves as perfectly rigid and the dependence on the megapacker traction $\Delta\sigma$ and the pore pressure $\Delta p*$ of the displacement vector, strain tensor and stress tensor fields on the EDZ is via the effective traction $(\Delta\sigma+\Delta p*)$ applied by the megapacker to the EDZ, in agreement with the considered generalisation of Terzaghi's effective stress principle.

A5. Weak form of the problem

A5.1. Balance of water mass

$$
\begin{aligned}
&\int_{[0,L]} W_w(z)\left\{\phi\frac{\partial}{\partial t}(\omega_l^w\rho_l S_l+\omega_g^w\rho_g S_g)+(\omega_l^w\rho_l S_l+\omega_g^w\rho_g S_g)\frac{\partial\varepsilon_{s,vol}}{\partial t}\right\}dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_w(z)\right)\left(\rho_l\left(\phi S_l D_l^w+d_l^l\frac{k_{rl}k}{\mu_l}\left|\frac{\partial p_l}{\partial z}\right|\right)\frac{\partial\omega_l^w}{\partial z}+\omega_l^w\rho_l\frac{k_{rl}k}{\mu_l}\frac{\partial p_l}{\partial z}\right)dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_w(z)\right)\left(\rho_g\left(\phi S_g D_g^w+d_g^l\frac{k_{rg}k}{\mu_g}\left|\frac{\partial p_g}{\partial z}\right|\right)\frac{\partial\omega_g^w}{\partial z}+\omega_g^w\rho_g\frac{k_{rg}k}{\mu_g}\frac{\partial p_g}{\partial z}\right)dz\\
&-W_w(L)\left\{\omega_l^w\rho_l\gamma_l\left(p_l^{ext}-p_l\right)+\beta_l\rho_l\left((\omega_l^w)^{ext}-\omega_l^w\right)\right\}\Big|_{z=L}\\
&-W_w(L)\left\{\omega_g^w\rho_g\gamma_g\left(p_g^{ext}-p_g\right)+\beta_g\rho_g\left((\omega_g^w)^{ext}-\omega_g^w\right)\right\}\Big|_{z=L}\\
&-W_w(0)\frac{M_w(t)}{\pi(r_e^2-r_i^2)}=0
\end{aligned}
$$

This expression is of the form

$$
\begin{aligned}
&\int_{[0,L]} W_w(z)\left\{A_w(p_l(z,t),p_g(z,t))\frac{d\sigma}{dt}(t)\right\}dz\\
&+\int_{[0,L]} W_w(z)\left\{B_w(p_l(z,t),p_g(z,t))\frac{\partial p_l}{\partial t}(z,t)+C_w(p_l(z,t),p_g(z,t))\frac{\partial p_g}{\partial t}(z,t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_w(z)\right)\left\{D_w(p_l(z,t),p_g(z,t))\frac{\partial p_l}{\partial z}(z,t)+E_w(p_l(z,t),p_g(z,t))\frac{\partial p_g}{\partial z}(z,t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_w(z)\right)\left\{F_w(p_l(z,t),p_g(z,t))\left|\frac{\partial p_l}{\partial z}\right|(z,t)\frac{\partial p_l}{\partial z}(z,t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_w(z)\right)\left\{G_w(p_l(z,t),p_g(z,t))\left|\frac{\partial p_l}{\partial z}\right|(z,t)\frac{\partial p_g}{\partial z}(z,t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_w(z)\right)\left\{H_w(p_l(z,t),p_g(z,t))\left|\frac{\partial p_g}{\partial z}\right|(z,t)\frac{\partial p_l}{\partial z}(z,t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_w(z)\right)\left\{I_w(p_l(z,t),p_g(z,t))\left|\frac{\partial p_g}{\partial z}\right|(z,t)\frac{\partial p_g}{\partial z}(z,t)\right\}dz\\
&+W_w(L)\left\{J_w(p_l(z,t),p_g(z,t))\right\}\Big|_{z=L}\\
&+W_w(L)\left\{K_w(p_l(z,t),p_g(z,t))p_l(z,t)\right\}\Big|_{z=L}\\
&+W_w(L)\left\{L_w(p_l(z,t),p_g(z,t))p_g(z,t)\right\}\Big|_{z=L}\\
&+W_w(0)\left\{\frac{-M_w(t)}{\pi(r_e^2-r_i^2)}\right\}=0
\end{aligned}
$$

where

$$A_w(p_l, p_g) = (\omega_l^w \rho_l S_l + \omega_g^w \rho_g S_g) 2A_{\Delta\sigma}$$

$$B_w(p_l, p_g) = \phi \frac{\partial}{\partial p_l}(\omega_l^w \rho_l S_l + \omega_g^w \rho_g S_g) + (\omega_l^w \rho_l S_l + \omega_g^w \rho_g S_g) 2A_{\Delta p*} \frac{\partial p*}{\partial p_l}$$

$$C_w(p_l, p_g) = \phi \frac{\partial}{\partial p_g}(\omega_l^w \rho_l S_l + \omega_g^w \rho_g S_g) + (\omega_l^w \rho_l S_l + \omega_g^w \rho_g S_g) 2A_{\Delta p*} \frac{\partial p*}{\partial p_g}$$

$$D_w(p_l, p_g) = \rho_l \phi S_l D_l^w \frac{\partial \omega_l^w}{\partial p_l} + \rho_g \phi S_g D_g^w \frac{\partial \omega_g^w}{\partial p_l} + \omega_l^w \rho_l \frac{k_{rl} k}{\mu_l}$$

$$E_w(p_l, p_g) = \rho_l \phi S_l D_l^w \frac{\partial \omega_l^w}{\partial p_g} + \rho_g \phi S_g D_g^w \frac{\partial \omega_g^w}{\partial p_g} + \omega_g^w \rho_g \frac{k_{rg} k}{\mu_g}$$

$$F_w(p_l, p_g) = \rho_l d_l^l \frac{k_{rl} k}{\mu_l} \frac{\partial \omega_l^w}{\partial p_l}$$

$$G_w(p_l, p_g) = \rho_l d_l^l \frac{k_{rl} k}{\mu_l} \frac{\partial \omega_l^w}{\partial p_g}$$

$$I_w(p_l, p_g) = \rho_g d_g^l \frac{k_{rg} k}{\mu_g} \frac{\partial \omega_g^w}{\partial p_g}$$

$$J_w(p_l, p_g) = -\left\{\omega_l^w \rho_l \gamma_l p_l^{ext} + \beta_l \rho_l \left((\omega_l^w)^{ext} - \omega_l^w\right)\right\}$$
$$-\left\{\omega_g^w \rho_g \gamma_g p_g^{ext} + \beta_g \rho_g \left((\omega_g^w)^{ext} - \omega_g^w\right)\right\}$$

$$K_w(p_l, p_g) = \omega_l^w \rho_l \gamma_l$$

$$L_w(p_l, p_g) = \omega_g^w \rho_g \gamma_g$$

A5.2. Balance of air mass

$$
\begin{aligned}
&\int_{[0,L]} W_a(z)\left\{\phi\frac{\partial}{\partial t}(\omega_l^a\rho_l S_l+\omega_g^a\rho_g S_g)+(\omega_l^a\rho_l S_l+\omega_g^a\rho_g S_g)\frac{\partial\varepsilon_{s,vol}}{\partial t}\right\}dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_a(z)\right)\left(\rho_l\left(\phi S_l D_l^w+d_l^l\frac{k_{rl}k}{\mu_l}\left|\frac{\partial p_l}{\partial z}\right|\right)\frac{\partial\omega_l^a}{\partial z}+\omega_l^a\rho_l\frac{k_{rl}k}{\mu_l}\frac{\partial p_l}{\partial z}\right)dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_a(z)\right)\left(\rho_g\left(\phi S_g D_g^w+d_g^l\frac{k_{rg}k}{\mu_g}\left|\frac{\partial p_g}{\partial z}\right|\right)\frac{\partial\omega_g^a}{\partial z}+\omega_g^a\rho_g\frac{k_{rg}k}{\mu_g}\frac{\partial p_g}{\partial z}\right)dz\\
&-W_a(L)\left\{\omega_l^a\rho_l\gamma_l\left(p_l^{ext}-p_l\right)+\beta_l\rho_l\left((\omega_l^a)^{ext}-\omega_l^a\right)\right\}\Big|_{z=L}\\
&-W_a(L)\left\{\omega_g^a\rho_g\gamma_g\left(p_g^{ext}-p_g\right)+\beta_g\rho_g\left((\omega_g^a)^{ext}-\omega_g^a\right)\right\}\Big|_{z=L}\\
&-W_a(0)\frac{M_a(t)}{\pi(r_e^2-r_i^2)}=0
\end{aligned}
$$

This expression is of the form

$$
\begin{aligned}
&\int_{[0,L]} W_a(z)\left\{A_a(p_l(z,t),p_g(z,t))\frac{d\sigma}{dt}(t)\right\}dz\\
&+\int_{[0,L]} W_a(z)\left\{B_a(p_l(z,t),p_g(z,t))\frac{\partial p_l}{\partial t}(z,t)+C_a(p_l(z,t),p_g(z,t))\frac{\partial p_g}{\partial t}(z,t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_a(z)\right)\left\{D_a(p_l(z,t),p_g(z,t))\frac{\partial p_l}{\partial z}(z,t)+E_a(p_l(z,t),p_g(z,t))\frac{\partial p_g}{\partial z}(z,t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_a(z)\right)\left\{F_a(p_l(z,t),p_g(z,t))\left|\frac{\partial p_l}{\partial z}\right|(z,t)\frac{\partial p_l}{\partial z}(z,t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_a(z)\right)\left\{G_a(p_l(z,t),p_g(z,t))\left|\frac{\partial p_l}{\partial z}\right|(z,t)\frac{\partial p_g}{\partial z}(z,t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_a(z)\right)\left\{H_a(p_l(z,t),p_g(z,t))\left|\frac{\partial p_g}{\partial z}\right|(z,t)\frac{\partial p_l}{\partial z}(z,t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{d}{dz}W_a(z)\right)\left\{I_a(p_l(z,t),p_g(z,t))\left|\frac{\partial p_g}{\partial z}\right|(z,t)\frac{\partial p_g}{\partial z}(z,t)\right\}dz\\
&+W_a(L)\left\{J_a(p_l(L,t),p_g(L,t))\right\}\\
&+W_a(L)\left\{K_w(p_l(z,t),p_g(z,t))p_l(z,t)\right\}\Big|_{z=L}\\
&+W_a(L)\left\{L_w(p_l(z,t),p_g(z,t))p_g(z,t)\right\}\Big|_{z=L}\\
&+W_a(0)\left\{\frac{-M_a(t)}{\pi(r_e^2-r_i^2)}\right\}=0
\end{aligned}
$$

where

$$A_a(p_l, p_g) = (\omega_l^a \rho_l S_l + \omega_g^a \rho_g S_g) 2A_{\Delta\sigma}$$

$$B_a(p_l, p_g) = \phi \frac{\partial}{\partial p_l}(\omega_l^a \rho_l S_l + \omega_g^a \rho_g S_g) + (\omega_l^a \rho_l S_l + \omega_g^a \rho_g S_g) 2A_{\Delta p*} \frac{\partial p*}{\partial p_l}$$

$$C_a(p_l, p_g) = \phi \frac{\partial}{\partial p_g}(\omega_l^a \rho_l S_l + \omega_g^a \rho_g S_g) + (\omega_l^a \rho_l S_l + \omega_g^a \rho_g S_g) 2A_{\Delta p*} \frac{\partial p*}{\partial p_g}$$

$$D_a(p_l, p_g) = \rho_l \phi S_l D_l^w \frac{\partial \omega_l^a}{\partial p_l} + \rho_g \phi S_g D_g^w \frac{\partial \omega_g^a}{\partial p_l} + \omega_l^a \rho_l \frac{k_{rl} k}{\mu_l}$$

$$E_a(p_l, p_g) = \rho_l \phi S_l D_l^w \frac{\partial \omega_l^a}{\partial p_g} + \rho_g \phi S_g D_g^w \frac{\partial \omega_g^a}{\partial p_g} + \omega_g^a \rho_g \frac{k_{rg} k}{\mu_g}$$

$$F_a(p_l, p_g) = \rho_l d_l^l \frac{k_{rl} k}{\mu_l} \frac{\partial \omega_l^a}{\partial p_l}$$

$$G_a(p_l, p_g) = \rho_l d_l^l \frac{k_{rl} k}{\mu_l} \frac{\partial \omega_l^a}{\partial p_g}$$

$$H_a(p_l, p_g) = \rho_g d_g^l \frac{k_{rg} k}{\mu_g} \frac{\partial \omega_g^a}{\partial p_l}$$

$$I_a(p_l, p_g) = \rho_g d_g^l \frac{k_{rg} k}{\mu_g} \frac{\partial \omega_g^a}{\partial p_g}$$

$$J_a(p_l, p_g) = -\left\{\omega_l^a \rho_l \gamma_l p_l^{ext} + \beta_l \left((\omega_l^a \rho_l)^{ext} - \omega_l^a \rho_l\right)\right\}$$
$$-\left\{\omega_g^a \rho_g \gamma_g p_g^{ext} + \beta_g \left((\omega_g^a \rho_g)^{ext} - \omega_g^a \rho_g\right)\right\}$$

$$K_a(p_l, p_g) = \omega_l^a \rho_l \gamma_l$$

$$L_a(p_l, p_g) = \omega_g^a \rho_g \gamma_g$$

A6. Shape functions, their derivatives and some related integrals

If the nodes on the domain $[0, L]$ are located at z^I ($I = 1, \ldots, N$, $z^1 = 0$, $z^N = L$), and we denote by $w^I(z)$ the shape function associated with node I ($I = 1, \ldots, N$), then

A6.1. Shape functions

The N linear two-noded shape functions associated with the N nodes read

$$w^1(z) = \begin{cases} \dfrac{z - z^2}{z^1 - z^2} & z^1 \le z \le z^2 \\ 0 & z^2 \le z \le z^N \end{cases}$$

$$w^I(z) = \begin{cases} 0 & z^1 \le z \le z^{I-1} \\ \dfrac{z - z^{I-1}}{z^I - z^{I-1}} & z^{I-1} \le z \le z^I \\ \dfrac{z - z^{I+1}}{z^I - z^{I+1}} & z^I \le z \le z^{I+1} \\ 0 & z^{I+1} \le z \le z^N \end{cases} \qquad (1 < I < N)$$

$$w^N(z) = \begin{cases} 0 & z^1 \le z \le z^{N-1} \\ \dfrac{z - z^{N-1}}{z^N - z^{N-1}} & z^{N-1} \le z \le z^N \end{cases}$$

Clearly, they are continuous, satisfy the conditions $w^I(z^J) = \delta^{IJ}$ ($I, J = 1, \ldots, N$) and are linear on each finite element $[z^I, z^{I+1}]$ ($I = 1, \ldots, N-1$).

A6.2. Derivatives of the shape functions

$$\frac{dw^1}{dz}(z) = \begin{cases} \dfrac{1}{z^1 - z^2} & z^1 \le z \le z^2 \\ 0 & z^2 \le z \le z^N \end{cases}$$

$$\frac{dw^I}{dz}(z) = \begin{cases} 0 & z^1 \le z \le z^{I-1} \\ \dfrac{1}{z^I - z^{I-1}} & z^{I-1} \le z \le z^I \\ \dfrac{1}{z^I - z^{I+1}} & z^I \le z \le z^{I+1} \\ 0 & z^{I+1} \le z \le z^N \end{cases} \qquad (1 < I < N)$$

$$\frac{dw^N}{dz}(z) = \begin{cases} 0 & z^1 \le z \le z^{N-1} \\ \dfrac{1}{z^N - z^{N-1}} & z^{N-1} \le z \le z^N \end{cases}$$

A6.3. Integrals of the shape functions and of their binary products

$$\int_{[0,L]} w^1(z)dz = \frac{z^2 - z^1}{2}$$

$$\int_{[0,L]} w^I(z)dz = \frac{z^{I+1} - z^{I-1}}{2} = \frac{z^{I+1} - z^I}{2} + \frac{z^I - z^{I-1}}{2} \qquad (1 < I < N)$$

$$\int_{[0,L]} w^N(z)dz = \frac{z^N - z^{N-1}}{2}$$

$$\int_{[0,L]} w^1(z)w^1(z)dz = \frac{z^2 - z^1}{3}$$

$$\int_{[0,L]} w^I(z)w^I(z)dz = \frac{z^{I+1} - z^{I-1}}{3} = \frac{z^{I+1} - z^I}{3} + \frac{z^I - z^{I-1}}{3} \qquad (1 < I < N)$$

$$\int_{[0,L]} w^N(z)w^N(z)dz = \frac{z^N - z^{N-1}}{3}$$

$$\int_{[0,L]} w^I(z)w^{I+1}(z)dz = \frac{z^{I+1} - z^I}{6} \qquad (1 \le I \le N-1)$$

A7. Nodal Equations

A7.1. Balance of water mass

$$
\begin{aligned}
&\int_{[0,L]} w^I(z)\left\{A_w(\textstyle\sum_K w^K(z)p_l^K(t),\sum_L w^L(z)p_g^L(t))\frac{d\sigma}{dt}(t)\right\}dz\\
&+\int_{[0,L]} w^I(z)\left\{B_w(\textstyle\sum_K w^K(z)p_l^K(t),\sum_L w^L(z)p_g^L(t))\sum_J w^J(z)\frac{dp_l^J}{dt}(t)\right\}dz\\
&+\int_{[0,L]} w^I(z)\left\{C_w(\textstyle\sum_K w^K(z)p_l^K(t),\sum_L w^L(z)p_g^L(t))\sum_J w^J(z)\frac{dp_g^J}{dt}(t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{dw^I}{dz}(z)\right)\left\{D_w(\textstyle\sum_K w^K(z)p_l^K(t),\sum_L w^L(z)p_g^L(t))\sum_J \frac{dw^J}{dz}(z)p_l^J(t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{dw^I}{dz}(z)\right)\left\{E_w(\textstyle\sum_K w^K(z)p_l^K(t),\sum_L w^L(z)p_g^L(t))\sum_J \frac{dw^J}{dz}(z)p_g^J(t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{dw^I}{dz}(z)\right)\times\\
&\quad\left\{F_w(\textstyle\sum_K w^K(z)p_l^K(t),\sum_L w^L(z)p_g^L(t))\left|\sum_J \frac{dw^M}{dz}(z)p_l^M(t)\right|\sum_M \frac{dw^J}{dz}(z)p_l^J(t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{dw^I}{dz}(z)\right)\times\\
&\quad\left\{G_w(\textstyle\sum_K w^K(z)p_l^K(t),\sum_L w^L(z)p_g^L(t))\left|\sum_J \frac{dw^M}{dz}(z)p_l^M(t)\right|\sum_M \frac{dw^J}{dz}(z)p_g^J(t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{dw^I}{dz}(z)\right)\times\\
&\quad\left\{H_w(\textstyle\sum_K w^K(z)p_l^K(t),\sum_L w^L(z)p_g^L(t))\left|\sum_J \frac{dw^M}{dz}(z)p_g^M(t)\right|\sum_M \frac{dw^J}{dz}(z)p_l^J(t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{dw^I}{dz}(z)\right)\times\\
&\quad\left\{I_w(\textstyle\sum_K w^K(z)p_l^K(t),\sum_L w^L(z)p_g^L(t))\left|\sum_J \frac{dw^M}{dz}(z)p_g^M(t)\right|\sum_M \frac{dw^J}{dz}(z)p_g^J(t)\right\}dz\\
&+w^I(L)\left\{J_w(\textstyle\sum_K w^K(L)p_l^K(t),\sum_L w^L(L)p_g^L(t))\right\}\\
&+w^I(L)\left\{\delta_I^N K_w(\textstyle\sum_K w^K(L)p_l^K(t),\sum_L w^L(L)p_g^L(t))\right\}p_l^N(t)\\
&+w^I(L)\left\{\delta_I^N L_w(\textstyle\sum_K w^K(L)p_l^K(t),\sum_L w^L(L)p_g^L(t))\right\}p_g^N(t)\\
&+w^I(0)\left\{\frac{-M_w(t)}{\pi(r_e^2-r_i^2)}\right\}=0
\end{aligned}
$$

A7.2. Balance of air mass

$$
\begin{aligned}
&\int_{[0,L]} w^I(z)\left\{A_a(\textstyle\sum_K w^K(z)p_l^K(t),\sum_L w^L(z)p_g^L(t))\frac{d\sigma}{dt}(t)\right\}dz\\
&+\int_{[0,L]} w^I(z)\left\{B_a(\textstyle\sum_K w^K(z)p_l^K(t),\sum_L w^L(z)p_g^L(t))\sum_J w^J(z)\frac{dp_l^J}{dt}(t)\right\}dz\\
&+\int_{[0,L]} w^I(z)\left\{C_a(\textstyle\sum_K w^K(z)p_l^K(t),\sum_L w^L(z)p_g^L(t))\sum_J w^J(z)\frac{dp_g^J}{dt}(t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{dw^I}{dz}(z)\right)\left\{D_a(\textstyle\sum_K w^K(z)p_l^K(t),\sum_L w^L(z)p_g^L(t))\sum_J \frac{dw^J}{dz}(z)p_l^J(t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{dw^I}{dz}(z)\right)\left\{E_a(\textstyle\sum_K w^K(z)p_l^K(t),\sum_L w^L(z)p_g^L(t))\sum_J \frac{dw^J}{dz}(z)p_g^J(t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{dw^I}{dz}(z)\right)\times\\
&\quad\left\{F_a(\textstyle\sum_K w^K(z)p_l^K(t),\sum_L w^L(z)p_g^L(t))\left|\sum_J \frac{dw^M}{dz}(z)p_l^M(t)\right|\sum_M \frac{dw^J}{dz}(z)p_l^J(t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{dw^I}{dz}(z)\right)\times\\
&\quad\left\{G_a(\textstyle\sum_K w^K(z)p_l^K(t),\sum_L w^L(z)p_g^L(t))\left|\sum_J \frac{dw^M}{dz}(z)p_l^M(t)\right|\sum_M \frac{dw^J}{dz}(z)p_g^J(t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{dw^I}{dz}(z)\right)\times\\
&\quad\left\{H_a(\textstyle\sum_K w^K(z)p_l^K(t),\sum_L w^L(z)p_g^L(t))\left|\sum_J \frac{dw^M}{dz}(z)p_g^M(t)\right|\sum_M \frac{dw^J}{dz}(z)p_l^J(t)\right\}dz\\
&+\int_{[0,L]}\left(\frac{dw^I}{dz}(z)\right)\times\\
&\quad\left\{I_a(\textstyle\sum_K w^K(z)p_l^K(t),\sum_L w^L(z)p_g^L(t))\left|\sum_J \frac{dw^M}{dz}(z)p_g^M(t)\right|\sum_M \frac{dw^J}{dz}(z)p_g^J(t)\right\}dz\\
&+w^I(L)\left\{J_a(\textstyle\sum_K w^K(L)p_l^K(t),\sum_L w^J(L)p_g^J(t))\right\}\\
&+w^I(L)\left\{\delta_I^N K_a(\textstyle\sum_K w^K(L)p_l^K(t),\sum_L w^L(L)p_g^L(t))\right\}p_l^N(t)\\
&+w^I(L)\left\{\delta_I^N L_a(\textstyle\sum_K w^K(L)p_l^K(t),\sum_L w^L(L)p_g^L(t))\right\}p_g^N(t)\\
&+w^I(0)\left\{\frac{-M_a(t)}{\pi(r_e^2-r_i^2)}\right\}=0
\end{aligned}
$$

A8. Element and node matrices and vectors

A8.1. Matrix $\boldsymbol{C}_{[z^I, z^{I+1}]}$

The elements of the matrix $\boldsymbol{C}_{[z^I, z^{I+1}]}$ corresponding to the element $[z^I, z^{I+1}]$ are

$$C_{w,I}^{I,l}[p_l^K(t), p_g^L(t)] = \int_{[z^I, z^{I+1}]} w^I(z) w^I(z) B_w[p_l(z,t), p_g(z,t)] dz$$

$$C_{w,I}^{I,g}[p_l^K(t), p_g^L(t)] = \int_{[z^I, z^{I+1}]} w^I(z) w^I(z) C_w[p_l(z,t), p_g(z,t)] dz$$

$$C_{w,I+1}^{I,l}[p_l^K(t), p_g^L(t)] = \int_{[z^I, z^{I+1}]} w^I(z) w^{I+1}(z) B_w[p_l(z,t), p_g(z,t)] dz$$

$$C_{w,I+1}^{I,g}[p_l^K(t), p_g^L(t)] = \int_{[z^I, z^{I+1}]} w^I(z) w^{I+1}(z) C_w[p_l(z,t), p_g(z,t)] dz$$

$$C_{a,I}^{I,l}[p_l^K(t), p_g^L(t)] = \int_{[z^I, z^{I+1}]} w^I(z) w^I(z) B_a[p_l(z,t), p_g(z,t)] dz$$

$$C_{a,I}^{I,g}[p_l^K(t), p_g^L(t)] = \int_{[z^I, z^{I+1}]} w^I(z) w^I(z) C_a[p_l(z,t), p_g(z,t)] dz$$

$$C_{a,I+1}^{I,l}[p_l^K(t), p_g^L(t)] = \int_{[z^I, z^{I+1}]} w^I(z) w^{I+1}(z) B_a[p_l(z,t), p_g(z,t)] dz$$

$$C_{a,I+1}^{I,g}[p_l^K(t), p_g^L(t)] = \int_{[z^I, z^{I+1}]} w^I(z) w^{I+1}(z) C_a[p_l(z,t), p_g(z,t)] dz$$

$$C_{w,I}^{I+1,l}[p_l^K(t), p_g^L(t)] = \int_{[z^I, z^{I+1}]} w^{I+1}(z) w^I(z) B_w[p_l(z,t), p_g(z,t)] dz$$

$$C_{w,I}^{I+1,g}[p_l^K(t), p_g^L(t)] = \int_{[z^I, z^{I+1}]} w^{I+1}(z) w^I(z) C_w[p_l(z,t), p_g(z,t)] dz$$

$$C_{w,I+1}^{I+1,l}[p_l^K(t), p_g^L(t)] = \int_{[z^I, z^{I+1}]} w^{I+1}(z) w^{I+1}(z) B_w[p_l(z,t), p_g(z,t)] dz$$

$$C_{w,I+1}^{I+1,g}[p_l^K(t), p_g^L(t)] = \int_{[z^I, z^{I+1}]} w^{I+1}(z) w^{I+1}(z) C_w[p_l(z,t), p_g(z,t)] dz$$

$$C_{a,I}^{I+1,l}[p_l^K(t), p_g^L(t)] = \int_{[z^I, z^{I+1}]} w^{I+1}(z) w^I(z) B_a[p_l(z,t), p_g(z,t)] dz$$

$$C_{a,I}^{I+1,g}[p_l^K(t), p_g^L(t)] = \int_{[z^I, z^{I+1}]} w^{I+1}(z) w^I(z) C_a[p_l(z,t), p_g(z,t)] dz$$

$$C_{a,I+1}^{I+1,l}[p_l^K(t), p_g^L(t)] = \int_{[z^I, z^{I+1}]} w^{I+1}(z) w^{I+1}(z) B_a[p_l(z,t), p_g(z,t)] dz$$

$$C_{a,I+1}^{I+1,g}[p_l^K(t), p_g^L(t)] = \int_{[z^I, z^{I+1}]} w^{I+1}(z) w^{I+1}(z) C_a[p_l(z,t), p_g(z,t)] dz$$

A8.2. Matrix $\boldsymbol{K}_{[z^I,z^{I+1}]}$

The elements of the matrix $\boldsymbol{K}_{[z^I,z^{I+1}]}$ corresponding to the element $[z^I, z^{I+1}]$ are (1/2)

$$K_{w,I}^{I,l}[p_l^K(t), p_g^L(t)] = \frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]} D_w[p_l(z,t), p_g(z,t)]dz$$
$$+\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]}\left(F_w[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_l}{\partial z}(z,t)\right| + H_w[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_g}{\partial z}(z,t)\right|\right)dz$$
$$K_{w,I}^{I,g}[p_l^K(t), p_g^L(t)] = \frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]} E_w[p_l(z,t), p_g(z,t)]dz$$
$$+\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]}\left(G_w[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_l}{\partial z}(z,t)\right| + I_w[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_g}{\partial z}(z,t)\right|\right)dz$$
$$K_{w,I+1}^{I,l}[p_l^K(t), p_g^L(t)] = -\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]} D_w[p_l(z,t), p_g(z,t)]dz$$
$$-\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]}\left(F_w[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_l}{\partial z}(z,t)\right| + H_w[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_g}{\partial z}(z,t)\right|\right)dz$$
$$K_{w,I+1}^{I,g}[p_l^K(t), p_g^L(t)] = -\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]} E_w[p_l(z,t), p_g(z,t)]dz$$
$$-\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]}\left(G_w[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_l}{\partial z}(z,t)\right| + I_w[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_g}{\partial z}(z,t)\right|\right)dz$$

$$K_{a,I}^{I,l}[p_l^K(t), p_g^L(t)] = \frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]} D_a[p_l(z,t), p_g(z,t)]dz$$
$$+\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]}\left(F_a[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_l}{\partial z}(z,t)\right| + H_a[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_g}{\partial z}(z,t)\right|\right)dz$$
$$K_{a,I}^{I,g}[p_l^K(t), p_g^L(t)] = \frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]} E_a[p_l(z,t), p_g(z,t)]dz$$
$$+\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]}\left(G_a[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_l}{\partial z}(z,t)\right| + I_a[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_g}{\partial z}(z,t)\right|\right)dz$$
$$K_{a,I+1}^{I,l}[p_l^K(t), p_g^L(t)] = -\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]} D_a[p_l(z,t), p_g(z,t)]dz$$
$$-\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]}\left(F_a[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_l}{\partial z}(z,t)\right| + H_a[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_g}{\partial z}(z,t)\right|\right)dz$$
$$K_{a,I+1}^{I,g}[p_l^K(t), p_g^L(t)] = -\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]} E_a[p_l(z,t), p_g(z,t)]dz$$
$$-\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]}\left(G_a[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_l}{\partial z}(z,t)\right| + I_a[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_g}{\partial z}(z,t)\right|\right)dz$$

The elements of the matrix $\boldsymbol{K}_{[z^I, z^{I+1}]}$ corresponding to the element $[z^I, z^{I+1}]$ are (2/2)

$$K_{w,I}^{I+1,l}[p_l^K(t), p_g^L(t)] = -\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]} D_w[p_l(z,t), p_g(z,t)]dz$$
$$-\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]}\left(F_w[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_l}{\partial z}(z,t)\right| + H_w[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_g}{\partial z}(z,t)\right|\right)dz$$

$$K_{w,I}^{I+1,g}[p_l^K(t), p_g^L(t)] = -\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]} E_w[p_l(z,t), p_g(z,t)]dz$$
$$-\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]}\left(G_w[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_l}{\partial z}(z,t)\right| + I_w[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_g}{\partial z}(z,t)\right|\right)dz$$

$$K_{w,I+1}^{I+1,l}[p_l^K(t), p_g^L(t)] = +\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]} D_w[p_l(z,t), p_g(z,t)]dz$$
$$+\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]}\left(F_w[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_l}{\partial z}(z,t)\right| + H_w[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_g}{\partial z}(z,t)\right|\right)dz$$

$$K_{w,I+1}^{I+1,g}[p_l^K(t), p_g^L(t)] = \frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]} E_w[p_l(z,t), p_g(z,t)]dz$$
$$+\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]}\left(G_w[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_l}{\partial z}(z,t)\right| + I_w[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_g}{\partial z}(z,t)\right|\right)dz$$

$$K_{a,I}^{I+1,l}[p_l^K(t), p_g^L(t)] = -\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]} D_a[p_l(z,t), p_g(z,t)]dz$$
$$-\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]}\left(F_a[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_l}{\partial z}(z,t)\right| + H_a[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_g}{\partial z}(z,t)\right|\right)dz$$

$$K_{a,I}^{I+1,g}[p_l^K(t), p_g^L(t)] = -\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]} E_a[p_l(z,t), p_g(z,t)]dz$$
$$-\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]}\left(G_a[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_l}{\partial z}(z,t)\right| + I_a[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_g}{\partial z}(z,t)\right|\right)dz$$

$$K_{a,I+1}^{I+1,l}[p_l^K(t), p_g^L(t)] = +\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]} D_a[p_l(z,t), p_g(z,t)]dz$$
$$+\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]}\left(F_a[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_l}{\partial z}(z,t)\right| + H_a[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_g}{\partial z}(z,t)\right|\right)dz$$

$$K_{a,I+1}^{I+1,g}[p_l^K(t), p_g^L(t)] = \frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]} E_a[p_l(z,t), p_g(z,t)]dz$$
$$+\frac{1}{(z^{I+1}-z^I)^2}\int_{[z^I,z^{I+1}]}\left(G_a[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_l}{\partial z}(z,t)\right| + I_a[p_l(z,t), p_g(z,t)]\left|\frac{\partial p_g}{\partial z}(z,t)\right|\right)dz$$

A8.3. Vector $\boldsymbol{g}_{[z^I, z^{I+1}]}$

The elements of the vector $\boldsymbol{g}_{[z^I, z^{I+1}]}$ corresponding to the element $[z^I, z^{I+1}]$ are

$$g_w^I[p_l^K(t), p_g^L(t), t] = -\int_{[z^I, z^{I+1}]} w^I(z) A_w[p_l(z,t), p_g(z,t)] \frac{d\sigma}{dt}(t) dz$$
$$g_a^I[p_l^K(t), p_g^L(t), t] = -\int_{[z^I, z^{I+1}]} w^I(z) A_a[p_l(z,t), p_g(z,t)] \frac{d\sigma}{dt}(t) dz$$
$$g_w^{I+1}[p_l^K(t), p_g^L(t), t] = -\int_{[z^I, z^{I+1}]} w^{I+1}(z) A_w[p_l(z,t), p_g(z,t)] \frac{d\sigma}{dt}(t) dz$$
$$g_a^{I+1}[p_l^K(t), p_g^L(t), t] = -\int_{[z^I, z^{I+1}]} w^{I+1}(z) A_a[p_l(z,t), p_g(z,t)] \frac{d\sigma}{dt}(t) dz$$

A8.4. Matrix $\boldsymbol{K}_{z^N}$

The elements of the matrix $\boldsymbol{K}_{z^N}$ corresponding to the node z^N are

$$K_{w,N}^{N,l}[p_l^K(t), p_g^L(t)] = K_w[p_l(L,t), p_g(L,t)]$$
$$K_{w,N}^{N,g}[p_l^K(t), p_g^L(t)] = L_w[p_l(L,t), p_g(L,t)]$$
$$K_{a,N}^{N,l}[p_l^K(t), p_g^L(t)] = K_a[p_l(L,t), p_g(L,t)]$$
$$K_{a,N}^{N,g}[p_l^K(t), p_g^L(t)] = L_a[p_l(L,t), p_g(L,t)]$$

A8.5. Vector $\boldsymbol{g}_{z^1}$

The elements of the vector $\boldsymbol{g}_{z^1}$ corresponding to the node z^1 are

$$g_w^1[p_l^K(t), p_g^L(t), t] = \frac{M_w(t)}{\pi(r_e^2 - r_i^2)}$$
$$g_a^1[p_l^K(t), p_g^L(t), t] = \frac{M_a(t)}{\pi(r_e^2 - r_i^2)}$$

A8.6. Vector $\boldsymbol{g}_{z^N}$

The elements of the vector $\boldsymbol{g}_{z^N}$ corresponding to the node z^N are

$$g_w^N[p_l^K(t), p_g^L(t), t] = -J_w[p_l(L,t), p_g(L,t)]$$
$$g_a^N[p_l^K(t), p_g^L(t), t] = -J_a[p_l(L,t), p_g(L,t)]$$